Concepts of Programming Languages: A Unified Approach

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Chapter 1

Introduction to Programming Languages

1.1. Programming languages

On December 17, 1903, the Wright Brothers achieved something that had, at the time, eluded everyone else who had been brave enough to try it: controlled powered flight in a heavier-than-air vehicle. Their first flight lasted only twelve seconds, but it signified the beginning of a new era in aviation. They did not achieve their feat by having a more powerful engine than anybody else. Their engine was homemade, and would burn itself up if run for more than about fifteen minutes. They did not succeed merely because they had the biggest wings. They succeeded where others had failed (and survived where some others did not) largely because they recognized that the fundamental problem in powered flight was not so much power as control, and they put significant effort into solving the problem of how to keep the airplane flying where they wanted it to go.

With computers, we face a similar problem of control. Today's computers are capable of executing billions or trillions of basic instructions per second. Their speed of computation gives them fantastic potential as tools. But a computer is nothing without control over the instructions that it must execute, and in what order they must be done. That control comes from software.

Computer software is written in a variety of different programming languages, which come in several flavors. A language can be very closely tied to the architecture of the computer, as is an assembly language (a symbolic form of machine language), it can be loosely tied to the computer's architecture, as is the language C, or it can be almost completely independent of what a computer actually looks like, as is, for example, Haskell. The language that you choose strongly influences characteristics of the software that you write, including

- Ease of programming, and ease of subsequent modification of your software (which correlates with low development cost);
- Portability of your software to a variety of computers;
- Reliability of your software (freedom from mistakes);
- Efficiency of your software (including time and memory requirements).

When choosing a language, you must decide which of those characteristics is most important to you. You might find that choosing a language that is very easy to use leads to software that is unacceptably slow, or that uses too much memory. On the other hand, choosing a language that offers very high efficiency can tend to lead either to low reliability or high cost. Without care, you might well find yourself in a position analogous to the one that faced some of the Wright Brothers' competitors, with an airplane that flies very fast, straight into the ground.

This book discusses different kinds of programming languages, including their desirable and undesirable characteristics and how they relate to one another. But the focus is not strictly on the languages themselves, for a language is nothing if it cannot be used. A major aim of this book is to show the reader how and when to use the different languages and the features that they provide. How you solve a programming or algorithmic problem depends on the perspective and style that you adopt. This book includes material on problem solving from different perspectives, so that you can make practical use of different kinds of languages.

1.2. Languages, libraries and implementations

The focus of this book is on aspects of programming languages. Inevitably, some closely related issues come into play.

Some languages have extensive libraries of tools that can be a major practical factor in making a language choice for a particular project, since the library might provide critical tools that you need. But libraries are not our focus here. The level of library support depends largely on how popular a language has been and how much corporate effort has been put into developing tools, and you can imagine an extensivle library added to other languages just as well. What does interest us here are language features that make it easy or possible to provide versatile libraries. Indeed, much of the innovation in programming language design has been driven by the needs of libraries for versatility and generality. To illustrate, early versions of the language Pascal made it impossible to provide acceptably general support for operations involving arrays or matrices. Later versions made some corrections for that.

To be usable, a programming language needs the support of an implementation, including tools such as a compiler, an interpreter and possibly an editor, debugger and documentation manager. Often, all of the language implementation tools are combined into a single integrated development took or kit, and the tools provided by the kit can greatly ease practical issues of software development. But development kits are not our focus here. We look at some language implementation issues, partly because understanding how a program runs is helpful in understanding what the language provides to you and how to use it effectively.

1.3. General families of languages

Programming languages typically fall into two broad categories, *imperative languages* and *declarative languages*.

In an imperative language, the focus is on commands or actions that the program performs in a carefully controlled order. For historical reasons, a single command or a construct for controlling command order is called a *statement*. The imperative approach is

also called a *dynamic* approach since it views computation as a process in which things are constantly changing, and the intent of a program is to say how things change.

Two groups of imperative languages are the *procedural languages* and the *object-oriented languages*. A procedural language encourages the programmer to concentrate on algorithms and procedures that encapsulate algorithms, with flow of data among the procedures an important but secondary consideration. Pascal and C are examples of procedural languages,

An object-oriented language encourages programmers to think first about how data is organized and how information will flow in a program, with algorithms added later. C++, Java and Eiffel are examples of object-oriented languages.

In a declarative language, a program is a collection of facts. Since the program avoids talking about anything changing over time, the declarative approach is also called a *static* approach. But a compiler or interpreter converts the facts into actions that perform computation for you so that the dynamics are introduced behind the scenes.

Two kinds of declarative language are *functional languages* and *logic languages*. A key unit of a function language is an expression, such as $x+y$, and a fact is an equation indicating that two expressions have the same value. The language implementation computes values of expressions based in the facts. Haskell and Standard ML are examples of functional languages.

A logic language is based on properties of things, and facts are either basic properties or implications indicating that, if some collection of properties are all true then some other property is also true. For example, a basic fact might state that Rhonda is Larry's mother and an implication could state that, if a person X is person Y's mother, then X is also Y's parent. You ask the language implementation to prove or disprove a property, based on the facts. For example, the language implementation would be responsible for realizing, from the available facts, that Rhonda is Larry's parent. Prolog and Eclipse are examples of a logic programming languages.

Although some languages are careful to stay within just one style, most are really mixtures, with focus on one of the styles. An imperative program is mostly based on commands, although imperative languages usually also support expressions, which have a functional feel to them. But even when you write commands in an imperative program, you are thinking of facts that guide the way you write commands, and those fact lie in the program implicitly, below the surface. In a well-written program, the guiding facts are often written explicitly as comments.

Similarly, a declarative program must have, below the surface facts, a way of performing computation. Since computation is ultimately performed on a computer that executes instructions, there are commands lying beneath the surface of a declarative program. The distinction between the imperative and declarative languages lies in what is clearly visible in programs: commands or facts. In a declarative program, the program itself consists of facts, and the rules for evaluating programs are beneath the surface. In an imperative program, the commands are clearly visible, and the facts are hidden behind them.

1.4. Encapsulations

Programs are not created and left alone. A program that is in use is almost always also being modified and improved, and programmers spend most of their time modifying existing software. So it makes sense for a programming language to provide features that make

programs easier to modify.

One way to do that is to provide mechanisms for *encapsulation*. Generally, encapsulation refers to any way of hiding some aspects of something. An encapsulation is a barrier beyond which certain modifications do not propagate.

Functions (also called procedures, methods or subprograms) are an obvious encapsulation mechanism. A subprogram hides how it works; any modification to the subprogram that does not affect what it accomplishes cannot affect any other part of the program, and so cannot propagate out into the rest of the program. Encapsulations are also called *abstractions* and a function or procedure is an example of a *procedural abstraction*, since it hides procedural details.

A type or class is an encapsulation that describes a particular kind of data value or object, hiding details of how the information within the value or object is stored. For example, suppose that a program creates a type Stopwatch, where you can start and stop a Stopwatch, and can ask it for the elapsed time. You use a Stopwatch via procedures and functions. For example,

```
Stopwatch w; // w is a Stopwatch.
start(w);
...
stop(w);
t = elapsedTime(w);
```

creates and uses a Stopwatch. The type hides how the information about the start and stop time are implemented. It also includes procedural abstraction, hiding how start, stop and elapsedTime work. If you modify how the information is stored, you might need to change how start, stop and elapsedTime work, but you certainly will not need to change any other part of the program. Types and classes are called *data abstractions*.

1.4.1. Modules

You typically break a large program into a collection of *modules*, each defining a collection of things such as constants, subprograms and types. But only some of those are available to other modules, the others hidden within the module. Any modification to hidden components that does not affect what the visible components do cannot propagate outside the module.

Each module typically exercises two forms of control over visibility. *Export control* limits which things that are defined in this module can be used in other modules, and *import control* limits which things from other modules are visible in this module.

Exports are usually controlled by one of two mechanisms. One is to put everything in one section or file, but to mark each thing either public or private. Java is an example of a language that takes that approach. The other is to separate a module into two sections (sometimes even two different files), one section telling what is exported, without including hidden details, such as how a function works, and the other defining hidden things as well as details of implementations of exported things. That is the approach taken by languages such as Cinnameg and Ada. Each approach has advantages. Java's approach, for example, allows you to write everything about a particular thing, such as a method, in one place. That eases modifications. But, by the same token, it means that, if you want to read only about what is exported, you find your reading cluttered with implementation details that you have no interest in and that you really should not see at all. Separating a module into

two sections, while forcing some modifications to be made in both sections, also avoids the clutter.

It is not a good idea to give a module access to things that it does not need. You do not want a misspelling of a name to wind up using a thing that you do not even know exists. Also, if you define something called "rock", you probably do not want to be burdened by the fact that some other module that you do not even use also has something called rock. A module typically only imports those other modules that it really needs. Some languages provide for close control over what gets imported. In Cinnameg, for example,

```
Import select, selectSuffix from "collect/search".
```

imports only functions select and selectSuffix from module collect/search.

1.4.2. Namespaces

Large libraries consist of many modules, and can be built up over time by the accumulated contributions of different authors. It is entirely possible, even likely, that two different modules each use the same name, for different things. What happens if module A wants to use functions called getSize from both modules B and C (for different purposes)?

Namespaces offer a solution. Each name has two parts: a namespace, indicating the module from which it comes; and its name within the module. For example, something called fruit in module cake might be called cake.fruit, but a thing called fruit in module salad would be called salad.fruit. Namespaces can be used as a form import control. Just mentioning cake.fruit is an implicit request to import fruit from module cake. That is how imports are controlled in Java.

But namespaces are not required to be tied to modules. They can represent logical modules that are spread across different physical modules. In Java, for example, a namespace is associated with a *package*, which is a collection of modules. In Cinnameg any physical module can choose which namespaces it contributes to.

Namespaces can be long, and can even have several parts. For example, to make Y be the square root of X in an Ada program, you write

```
Y := Ada.Numerics.Generic_Elementary_Functions.Sqrt(X);
```

That is just intolerable, and some way needs to be provided to shorten names. In Ada,

```
use Ada.Numerics.Generic_Elementary_Functions;
```

tells the compiler to replace Sqrt (and each other name with the listed namespace) by its long name, Ada.Numerics.Generic_Elementary_Functions.Sqrt. Java has a similar mechanism, where `import java.util.*` says to allow short names for all things in package java.util. If you only want to use Random from that package, directive `import java.util.Random`. imports only Random. Of course, anything else in package java.util is still available using its full name.

1.4.3. Intension and extension

Two useful ideas from logic are the ideas of *intension* and *extension*. The intension of a concept is what it means to you, in a conceptual and, preferably, permanent way. For example, the intension of the concept *the number of planets in the solar system* is just what it sounds like, the concept of the number of planets in the solar system.

The extension is the current value. The ancients counted five planets. (They did not count Earth as a planet, and did not know about planets beyond Saturn.) A few years ago, most people would have said that there are 9 planets. With the demotion of Pluto to a dwarf planet, that number has changed to 8.

The point is that extensions tend to change much more rapidly than intensions. So, to limit the effects of modifications, a programmer should work, wherever possible, in terms of intensions rather than in terms of extensions. That is where encapsulations come into play. The encapsulating concept is the intension. The current (encapsulated) definition is the extension.

For example, named constants are a simple form of encapsulation. Suppose that you define a constant in a C++ program by

```
const int maxCookies = 100;
```

so that you can only have 100 cookies at once. The intension is *maxCookies*, the maximum number of cookies that can be held at once, and the entire rest of the program refers to that without being concerned what its value is. The current extension is 100. Later, changing maxCookies to 200 does not require modifying any part of the program except the one line that defines maxCookies.

1.4.4. Language support

Of course, all of this only works in programs if the language supports it. If there is no way to define a named constant, you cannot create the encapsulation maxCookies.

Almost all languages provide some kind of procedural abstraction, and most provide support for modules. Of the encapsulations mentioned above, the most difficult to do well is data abstraction. For example, suppose that you want to create type Stopwatch in C. So you define a type, indicating that a Stopwatch carries with it the times at which it started and stopped (each of type *double*, indicating a real number in C).

```
struct Stopwatch {
   double startTime;
   double stopTime;
};
```

Now comes a problem. If you put this type definition where other modules can use it, those other modules can make direct use of startTime and stopTime; they are not hidden at all, and there is no real data abstraction. To hide startTime and stopTime, you are allowed to replace the definition of Stopwatch by

```
struct Stopwatch;
```

suppressing the details. Unfortunately, doing that makes it impossible for other modules to create a Stopwatch because the compiler does not know how much memory to allocate for one. In C, there is no good way to accomplish data abstraction.[1] C++, an extension of C, provides some ways to achieve genuine data abstraction via object-oriented programming.

[1] The usual approach is to use *pointers*. You are allowed to use a pointer to a Stopwatch without knowing how much memory an object of type Stopwatch uses. But even that is not full data abstraction because it forces modules to use explicit pointers and to know that they are using pointers, even when they would prefer to be unaware of that.

1.5. Exercises

1.1. What is the defining characteristic of a declarative programming language?

1.2. What is the defining characteristic of an imperative programming language?

1.3. Is a loop construct, such as a while loop, more likely to be found in a procedural program or a functional program? Why?

1.4. Although programming languages are typically intended to be general purpose tools, some are for special purposes or domains, and some people use the term *fourth generation language* for specialized languages. Find two examples of fourth generation languages.

1.5. Some languages are advertised as *prototyping languages*, intended for use in developing prototypes of projects for testing out ideas. Sometimes Perl and Python are used for prototyping. How would you expect a prototyping language to differ from a general purpose language?

1.6. How do encapsulations aid modification of programs?

1.7. What is procedural abstraction?

1.8. What is data abstraction?

1.9. Modules are a form of encapsulation. What does a module hide?

1.10. How is a DVD player an abstraction or encapsulation? What does it hide?

1.11. How is a publisher of books an encapsulation? What does it hide?

1.12. Where are programming language features typically put more to the test, in libraries or in application programs?

1.6. Bibliographic notes

Wall et. al. [98] describe the prototyping language Perl. Prolog is covered by Clocksin and Mellish [28], and Meyer [69] gives a detailed description of Eiffel. Bird and Wadler [15] and Thompson [95] explain how to use Haskell, and Paulson [78] covers functional programming using Standard ML.

Chapter 2

Some Programming Languages

This book draws examples from several programming languages. To get you ready, here are some brief background on those languages. We also draw many examples from a Cinnameg, a language designed for illustrating concepts of programming languages, and we end this chapter with a very brief introduction to Cinnameg.

This chapter is not intended to give anything like a full list of programming languages. Such a list would be nearly impossible since thousands of languages have been created. Instead, we restrict attention to a few that are mentioned in the book.

2.1. Fortran

Developed in the 1950s and heavily influenced initially by the architecture of the IBM 704 computer, Fortran was the first "high level" programming language. Fortran has evolved over the decades since then, It is still in use today, partly because of a large body of libraries for numerical computations that are available and partly because of its continued focus on supporting high performance computing.

Fortran was created at a time when assembly language was king and nobody had a good idea what another kind of language should look like. By present day standards, it has an unusual appearance. Looking at early versions of Fortran, you find that Fortran is line-oriented (one statement per line), with provisions for extending a statement to multiple lines. Lines are broken into columns, with labels written in columns 2–5 and the statement in columns 7–72, motivated by the media (punched cards) that were employed at the time. Fortran does not allow recursion.

Some of those deficiencies were fixed with major revisions. For example, Fortran 77 introduced structured programming constructs eand Fortran 90 introduced free-form programs, defined types and recursion. Fortran 2003 introduced object-oriented features and features for parallel processing, and, with a few revisions, is now known as Fortran 2008. We only use Fortran for a few examples, sticking with the language Fortran IV, from about 1961 and becoming a standard in 1966, which remained the de facto standard language for numerical computing for nearly two decades. Fortran IV offers some interesting examples, but it is worth noting that Fortran is alive and well and keeping up with new ideas.

2.2. Algol 60 and its descendants

Algol 60 (proposed in 1960, as the name suggests) did away with the line-oriented style of Fortran and adopted a more structured form, much closer to that of modern languages. It allows procedure definitions to be recursive.

Algol 60 has a certain elegance and simplicity. But it also has deficiencies. Like early Fortran, it does not have any provision for creating new types, instead relying on integers, real numbers and boolean values. Only stack allocation of data is available. Later languages extended Algol 60 to alleviate some of its weaknesses. Pascal, initially defined in 1970 and reported on fully in 1974, was deliberately kept small and simple, but provided for new types and for allocation using a heap. A revised version became an ISO standard in 1991. We look at early versions of Pascal, since they illustrate some of the issues that need to be addressed. Pascal became, for about two decades, the de facto standard language for teaching computer programming.

At about the same time as Pascal, Algol 68 was also developed, with the intent of being a large extension of Algol 60 with an elaborate type system. For various reasons, partly due to the difficulty of reading the official report, Algol 68 did not enjoy the popularity of Pascal.

Beginning in the early 1970s, the U.S. Department of Defense realized that it needed to do something to reduce the high cost of software development. No existing languages were found to offer everthing that the DoD. needed, and, in 1975, it embarked on a project that was to last for the next five years to develop a suitable language, eventually called Ada. Ada became an ANSI standard in 1983 and an ISO standard in 1987. The purpose of Ada was to reduce software cost, not to develop one more language, and to that end the DoD. sent the following memo in 1983.

> The Ada programming language shall become the single programming language for Defense mission-critical applications. Effective 1 January 1984 for programs entering Advanced Development and 1 July 1984 for programs entering Full-Scale Engineering Development, Ada shall be the programming language.

Ada was designed as an extension of Pascal. The 1983 version provides no support for object-oriented programming. One of the goals of the DoD. was to have a pure language, without subsets or supersets. That is, all compilers were required to accept exactly the same language, without extensions. But, beginning in 1988, that plan began to be abandoned as a new version, now called Ada 95, was developed to bring in object-oriented programming. Modifications to Ada continue to be worked on.

Ada has not lived up to its hopes as a universal programming language. The DoD. removed its requirement to use Ada in 1997.

2.3. C

The C language was designed in the 1970s as a typed version of an earlier typeless language, BCPL. It is intended for system programming and was used for an implementation of the Unix operating system. It became popular along with Unix because it was well supported by Unix. With a few revisions, C became an ANSI and ISO standard in 1990, with a second standard in 1999.

C offers a useful language for examples partly because, due to its popularity, much of its syntax was borrowed for more recent languages, so the syntax is likely to look familiar. C

is still heavily used for system development, acting as the closest thing to machine language that most programmers prefer to use.

2.4. Object-oriented languages

C provides no support for object-oriented programming, and its data abstraction capabilities are weak. C++ is an extension of C to object-oriented programming that is almost completely compatible with C. It was described in 1986 and underwent significant revisions over time, becoming an ISO standard in 1998. Revisions to the standard are still being made, including an update in 2011.

Smalltalk is a typeless object-oriented programming language whose idea was to take the idea of a class from Simula as far as it could be taken. It managed to do away with control structures such as loops and conditionals in the language by defining them in classes. Smalltalk went through revisions during the 1970s, with the 1980 version going into more widespread use than prior versions. One innovation of Smalltalk is an extensive facility for a program to view and modify its own implementation on the fly.

However, Smalltalk never enjoyed the popularity of Pascal or C, and its syntax is unusual. Java was introduced with the idea of being a typed version of Smalltalk, but with C syntax so that it could be learned easily by programmers familiar with C. Unlike C++, no attempt was made to make Java compatible with C, which would necessarily have taken it away from the Smalltalk model. C# takes a similar approach with Microsoft's .NET framework as the underlying model.

Eiffel is another example of a large object-oriented programming language. Eiffell has an intricate type system supporting multiple inheritance and is designed around software engineering principles for designing reliable software. We will not have much opportunity to comment on Eiffel, and Eiffel has not enjoyed the popularity of Java.

2.5. Functional languages

The foundations of functional programming began with the study of λ-calculus, a notation and formalism for talking about functions. Lisp combined λ-calculus with linked lists to yield a practical and simple programmming language. Variations on Lisp include Common Lisp, a large language, and Scheme, a much smaller language. All share a very simple syntax that is described in Chapter 11, and that makes it especially easy to convert between programs and data.

The language ML was designed in the 1970s to support development of automatic theorem provers (really proof checkers with capabilities of doing some proofs or parts of proofs on their own), including LCF and HOL. ML stands for Meta Language; logicians use that term for the language in which you discuss properties of another language. With significant revisions, ML was standardized in 1990, with an updated standard described in 1997; the standard language is called Standard ML, or simply SML.

SML is a strict, declarative functional programming language, but has some imperative features. One of its innovation is a polymorphic type system with type variables and full type inference.

Haskell is a nonstrict purely functional programming language. It has a polymorphic type system that has some similarity the SML's, but also with some capabilities similar to object-oriented languages. It was initially defined in 1990 and underwent revisions, with the

1998 version being popular. Haskell 2010 introduces the ability to use functions defined in other languages.

Haskell needs to provide a way of performing imperative operations, such as writing files, in a language that explicitly forbids them, a tall order. It accomplishes that partly by reversing the interaction between a program and an operating system. Instead of the programming making system calls, the usual model, a proxy for the operating system makes calls to the program to ask for instructions on what to do.

2.6. Prolog and its descendants

With the first implementation available in the early 1970s, Prolog is a programming language based on first order logic. Prolog grew out of work on logic and theorem proving, and its implemetation is based on two key ideas, unification and backtracking. A program is a collection of logical facts and the interpreter is a theorem prover that finds constructive proofs. Prolog demonstrates that all computation can be though of as constructive existence proofs, where the proof finds a value with a desired property.

There are quite a few versions of Prolog available. ISO Standard Prolog was defined in 1995, and is the version that we look at here.

2.7. Cinnameg

Cinnameg is a programming language that is used in this book to illustrate general programming language principles. The ideas presented by Cinnameg are representative of those found in a variety of other languages. Cinnameg is intended to be a conceptual programming language, and is further removed from the characteristics of computer hardware than some languages. For example, integers are of arbitrary size, not limited by the size of a machine word, so that the sum of two positive integers is always a positive integer, as it is in mathematics.

Some programming languages are *paradigmatic*, meaning that they enforce or highly encourage a particular programming paradigm. Java, for example, requires the programmer to adopt an object-oriented approach to programming; Prolog requires a logic-programming approach; and Haskell enforces a functional approach. Cinnameg is deliberately designed not to be paradigmatic, but to support a variety of paradigms. You can use Cinnameg either for declarative programming or for imperative programming. Much of the library is designed in a functional style because that tends to be conceptually the simplest and leads to the fewest errors. But different styles of programming can be blended seamlessly.

Some experts in programming languages believe that, wherever possible, a language should provide just one way to do a particular thing. That tends to reduce language complexity, makes the language easier to learn and can make programs easier to read. Because of Cinnameg's philosophy, that is not possible. The whole point is to provide a variety of ways of doing things. But Cinnameg is intended to be learned in subsets. You can begin using it without a large investment of time. Also, things that you have not yet seen are intended to be relatively easy to read and understand, as long as you understand the underlying ideas.

Here, we only discuss a few general issues that you should be aware of. Other issues are discussed where they are needed. You can find information at http://www.cs.ecu.edu/~karl/cinnameg/10-0/lin/.

Lexical issues

Cinnameg is free-form. In most contexts, a line break has the same meaning as a space. Normally, spaces are only allowed between tokens, and are only required where the boundary between two tokens would otherwise be unclear. A comment begins with %% and continues to the end of a line.

Identifiers have two forms. *Ordinary identifiers* begin with a letter and can contain letters, digits and question marks. For example, `horse1` and `nil?` are ordinary identifiers. Optionally, then can end on one or more apostrophes, as in `x'`. *Symbolic identifiers* are symbols such as +, *, /, ++ and +//. They can contain characters in string "`~#+|-*/^!=<>&$_\?@`", as long as they do not start with `?` or `'`. There are some reserved words and symbols are not allowed to be identifiers.

Showing structure

Most languages have a way to indicate that one thing is part of a larger one. Some languages use indentation (or *layout*) for that purpose. Because layout can lead to confusion, and interferes with a language being free-form, Cinnameg adopts an explicit syntax. You write `Ensure ...%Ensure`, beginning with an introductory word `Ensure` and ending with `%Ensure`, meaning end-of-Ensure. In general, any construct whose first word begins with an upper case letter must end on a corresponding end marker. If it starts with Foo, it ends with %Foo.

Explicit end-markers make it is clear where each unit of the program ends, but they are awkward for short things. To reduce the inconvenience, you can use a period as a replacement for the end marker. So you can write

```
Ensure x == y + 1.
```

rather than `Ensure x == y + 1 %Ensure`. When you write a period, ask yourself what it stands for. The begin word that it matches is always capitalized.

A semicolon ends one construct and begins another of the same kind. For example, when it occurs where %Ensure is expected, a semicolon stands for %Ensure Ensure, and

```
Ensure x == y + 1;
       z == w + 1.
```

is the same as

```
Ensure x == y + 1 %Ensure
Ensure z == w + 1 %Ensure
```

Some constructs, such as {`x = n+1`}, begin with a symbol, such as { and end with a matching symbol such as }.

2.8. Bibliographic notes

Backus [10] gives a history of Fortran, Sheridan [89] reports on an early version of Fortran, Ellis [37] describes Fortran 77 and Fortran 90, and Reid reports on Fortran 2003.

Ritchie [82] gives a history of C and Stroustrup [94] discusses the history of C++.

The Algol 60 Report (Naur [75]) describes Algol 60 and Ekman and Froberg offer an introduction.

Wirth [103] provides a history of Pascal, the Revised Pascal Report (Jensen and Wirth [57]) describes the language and Pascal 90 is described in [56].

Lindsey [66] gives a history of Algol 68, the Algol 68 report (van Wijngaarden et. al. [97]) gives a careful definition of Algol 68 and Lindsay and Van Der Muelen [67] give a more comprehensible introduction to Algol 68.

Whitaker [101] discusses the history of Ada and Ichbiah et. al. [54] offer a rationale for decisions made in the design. Barnes [13] is a text on Ada 95.

Nance [74] gives a broad description of simulation languages that includes Simula 67. Kay [58] gives an early history of Smalltalk and Goldberg and Robson [39] describe Smalltalk 80. Gosling, Joy, Steele and Bracha [43] give a specification of Java. Arnold, Gossling and Holmes [6] describe Java. Meyer [69] describes Eiffel.

Rosser [83] discusses the history of λ-calculus, the mathematical foundation of functional programming. Steele and Gabriel [92] give a history of Lisp. Hudak, Hughes, Jones and Wadler [53] discuss the history of Haskell. Milner, TOfte and Harper [71] give a careful definition of SML and Paulson [78] gives an introduction to programming in SML. Gordon [42] gives a history of LCF and HOL.

Colmerauer and Roussel [30] give a history of Prolog. Clocksin and Mellish [28] describe ISO Prolog.

Chapter 3

Syntax

3.1. Form and function

The *syntax* of a programming language tells what a program looks like, how a program is broken into words and symbols, how those words and symbols form larger units, and whether or not a given text document is a meaningful program. Syntax is what people tend to notice first about a language. Figure 3.1 shows the same thing written in three different programming languages, each with its own syntax. The differences between the Pascal and C forms are relatively small. For example, C uses a left brace where Pascal uses the word **begin**, and a right brace where Pascal uses **end**. In C you must write a semicolon after the last statement in a group, but in Pascal you do not. But syntax can vary a great deal, as is apparent from the Scheme form of the same thing.

Ideally, the appearance of the program reflects how the program is organized and what the program accomplishes. Many programming languages are *free form*, which means that a line break has the same meaning, in most contexts, as a space, that several spaces in a row are equivalent to one space, and that spaces are only required to be placed where they are needed to avoid misunderstanding. That gives programmers the freedom to write programs in a form that conveys structure and meaning. For example, the C program fragment in Figure 3.2(a) shows, through well chosen line breaks and indentation, which parts are done when $x < y$, as well as the fact that the two steps w = y and z = 0 are done

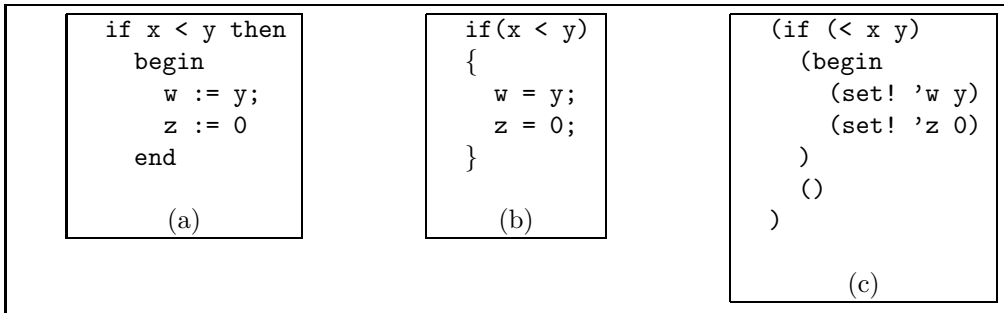


Figure 3.1: **A program fragment written in (a) Pascal, (b) C and (c) Scheme.**

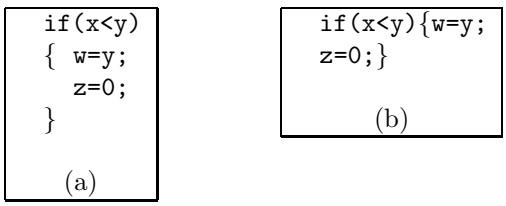


Figure 3.2: **Two layouts of a C fragment.**

one after the other. But a free-form language does not require a programmer to make form reflect function, and it is just a matter of taste how or whether to indent a program. Figure 3.2(b) shows the same program fragment in a less obvious form. Some languages prefer to make indentation mandatory. Haskell and Python use a form of syntax called *layout* in which indentation is a significant part of the meaning of a program. Haskell expression

```
let y = 2 * b
    f(x) = x + y
in f(c) * f(d)
```

computes $(c + 2 * b) * (d + 2 * b)$ by defining $y = 2 * b$ and function $f(x) = x + y = x + 2 * b$. It is indentation, not just line breaks, that signal the end of the first part ($y = 2 * b$) and the start of the next ($f(x) = x + y$).

There is much more than indentation to give a programmer hints about a program. Well chosen words help a lot. In Ada, a loop begins, naturally enough, with the word **loop**. A well-designed syntax has regular rules with few exceptions. It uses familiar notation where possible, and avoids notations that are misleading. The language C illustrates both good and bad choices in a single notational decision. To divide two numbers x and y, you write x/y in a C program, which is easy to read and remember because it is similar to familiar notation. But C also uses / to indicate division of integers, with the rule that, when m and n are integers, m/n is also an integer, obtained by dividing and ignoring any fractional part. In C, you create a variable that holds a real number by using the word **double** (odd choice?). Statement

```
double x = 2/3;
```

actually makes $x = 0.0$, because the division is between two integers 2 and 3. (The integer result, 0, is automatically converted to 0.0.) That certainly does not correspond with familiar notation. Ada solves that problem by using the word **div** to indicate integer division, at the cost of requiring the programmer to learn one more unfamiliar word.

Ideally, the syntax of a language should take into account that a programmer might make a mistake typing a program. A program that looks like it is saying one thing should not be easily transformed into something altogether different by the slip of a finger on a keyboard. Fortran offers an example of a poor choice in that respect. Figure 3.3 shows two Fortran program fragments that only differ in one place where a comma has been replaced by a period. Fragment (a) is a loop, but fragment (b) is a sequence of two statements.

Syntactic choices can influence how easy it is to learn and to use a language, and language designers give careful thought to syntax. Programmers, who have little choice but to use the syntax that is available, need to be aware of pitfalls, and to pay close attention to avoid them.

```
      DO 20 I = 1,100
20    X = F(X)
```

(a)

```
      DO 20 I = 1.100
20    X = F(X)
```

(b)

Figure 3.3: **Two Fortran program fragments.** Fragment (a) is a loop, performing statement X = F(X) 100 times. (The number 20 after DO is the label of the last statement in the loop body, and labels are written in the left margin.) Fragment (b) is a pair of assignment statements, the first setting variable DO20I to 1.100. Spaces are ignored in Fortran, and variables, such as DO20I, are created automatically when they are used. Precisely this kind of mistake was discovered by NASA in 1961 in a program that had been in use on spacecraft for navigational calculations, and had been carried on some Mercury orbiters. Fortunately, the mistake only reduced the precision of the calculations, and no disaster occurred as a result of it.

3.2. Describing syntax

Even after thinking through design decisions carefully, a language designer is faced with arbitrary choices concerning which words and symbols to use, their order, and other issues of program structure. At that point, the problem changes from one of choosing a "good" syntax to one of describing those arbitrary choices to people who need to use the language, in a way that programmers will find easy to understand.

Algol 60 is an early programming language. In 1958, John Backus, one of the principal designers of Algol 60, circulated a preliminary report describing the new language. To his surprise, he discovered from responses that, not only did readers not understand the meaning of programs, but they did not even understand what programs looked like. That is, they did not understand the rules for creating a syntactically correct program.

Backus solved the problem in an inspired way. The revised report included a clear and precise definition of the syntax of Algol 60, so that nobody could ever again be confused about the syntax. That worked so well that it has become standard practice in describing programming languages. The remaining sections show the methods that Backus employed for describing syntax, as well as a few extensions of those ideas.

3.2.1. Lexical rules

A program is just text, a sequence of characters. But the character level is too detailed for convenience. Instead, a program is broken into a sequence of basic words and symbols called *lexemes*. For example, C program fragment

```
while (turnip != 0) jump();
```

is thought of as a sequence of ten lexemes: **while**, **(**, **turnip**, **!=**, **0**, **)**, **jump**, **(**, **)** and **;**. Punctuation marks and special symbols are lexemes but spaces are not. Spaces, when they are present, just separate lexemes.

The first step in describing the syntax of a language is choice of a set of *lexical rules* that tell how a sequence of characters is broken up and converted to a sequence of lexemes. Typically, the language definition starts by saying what all of the lexemes are. For example,

the list of lexemes might start with reserved words such as **while** and **if**, as well as symbols such as % and ==. The form of an *identifier*, or name, is part of this list. For example, an identifier might be defined to be any sequence of one or more letters. Then a typical lexical rule is that you start at the beginning of the program, removing one lexeme at a time. To get the next lexeme, you first skip over anything, such as spaces, that does not belong to a lexeme. Then you take the maximum number of characters that make up a valid lexeme of the language. For example, if some number of lexemes have already been obtained, and the rest of the program is " `while(...`", then the next lexeme is **while**, not, for example, **whi**, which might itself be a valid lexeme (such as a variable name). The next lexeme after that is the left parenthesis, and lexemes continue to be obtained by the same rule until the entire program has been broken into lexemes.

3.2.1.1. Comments

Most languages allow programs to contain comments, which can typically be put almost anywhere in a program, and contribute nothing to the meaning of the program, as far as the language definition is concerned. Comments are similar to spaces, and are not parts of lexemes. For example, one of the lexical rules of C is that, when you are about to get the next lexeme, you should skip over anything that starts with `/*`, up to the next occurrence of character sequence `*/`.

3.2.1.2. Tokens

The fact that the lexeme **while** is spelled w-h-i-l-e is certainly important for writing programs, but once the lexemes have been obtained, the exact spelling is less important than the fact that **while** is the word that introduces a loop. It is natural to ignore the individual characters, and to think of **while** as an indivisible component of the language, called a *token*. A token is represented by a lexeme, but is not the same thing as a lexeme; how a word is spelled is a property of the lexeme, not of the token.

We have seen lexical rules that tell how to break a program into lexemes. There are additional lexical rules that give a correspondence between lexemes and tokens. For example, a lexical rule might state that lexeme **while** stands for the token **while**, or even the token **loopbegin**. You can call the tokens anything that you like. The remaining syntactic rules are expressed entirely in terms of tokens.

It is important to use tokens instead of lexemes in syntactic rules. At a given place in a program, you might be allowed to write any integer constant. But there are infinitely many integer constants, and you cannot list them all. You do not even want to say, in syntactic rules, what an integer constant looks like, since that is really a lexical issue. Instead, you have just one token, say **number**, that has infinitely many lexemes. The lexical rules can say that lexeme **342** stands for token **number**.

From a syntactic viewpoint, you probably do not care which integer constant is written in a program. Replacing one integer constant by another in a meaningful program should yield another meaningful program. But the meaning has changed, and to account for that, a token that has more than one lexeme usually has an *attribute* that tells the corresponding lexeme. For example, lexeme **95** might correspond to token **number** with attribute "95".

A side benefit of using tokens instead of lexemes is that it is easy to modify a language by changing the words and symbols. If you want a Norwegian version of Java, you can just say that the lexeme **hvis** (Norwegian for "if") stands for the token **if**, with no need to any other changes. That was more important in the past when computers were less standardized

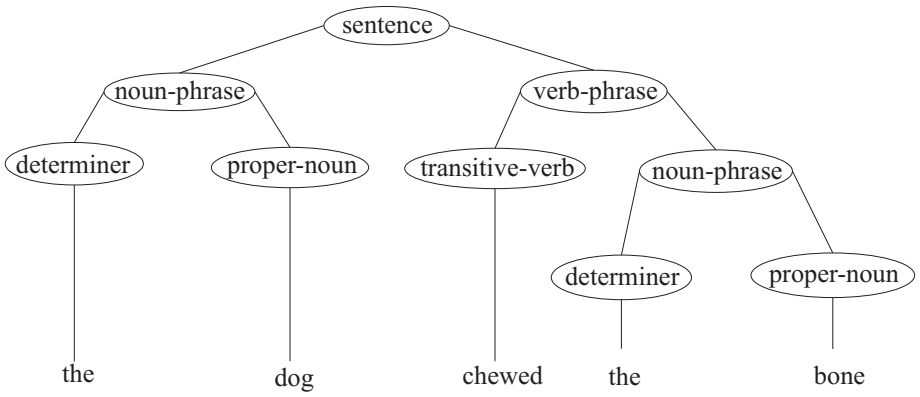


Figure 3.4: **The structure of an English sentence.** Notice that the words are leaves. Concepts such as verb phrase and noun phrase are used to label nonleaves.

than they are today. For example, Pascal indicates that token **pointer** can be written ↑. But few computer have an up-arrow character, so lexemes ^ and **ptr** also stand for the same token.

Although the distinction between lexemes and tokens is an important one, it can be awkward at times to make that distinction in practice. The most natural way of referring to the token **if** is by using its lexeme, **if**, as the token's name. When a token has just one lexeme, it is common practice to use the lexeme as the name for the token.

3.2.2. Program structure and parse trees

The structure of an English sentence can be described by a tree. For example, the structure of sentence "The dog chewed the bone" *might be* as shown in Figure 3.4. The structure of a program, or part of a program, can also be expressed as a tree. The internal nodes of the tree describe abstract units of a program, such as statements and expressions, and the leaves are labeled by tokens. Figure 3.5 shows a plausible structure for expression 4 + 9 * 2. You can read off the sequence of tokens that a tree describes by reading the leaves from left to right. Tree diagrams of programs are called *parse trees*, and the labels of nonleaf nodes are called *nonterminals*. Notice that nonterminals, such as expression, are not really part of the language, but are only part of the syntactic description of the language.

Some parse trees allow certain leaves to be labeled by nonterminals, indicating parts of the program that have no corresponding tokens. For example, Pascal allows a statement to be nothing at all, indicating that nothing should be done. Figure 3.6(a) shows a corresponding tree, which is just a leaf that is labeled **statement**. When you read off the sequence of tokens that a tree stands for, skip over leaves that do not contain tokens.

3.2.3. Using grammars to indicate allowed trees

To describe the syntax of a language, you describe exactly what the parse trees for that language look like. That way, not only are you saying which sequences of tokens are allowed (since they are just the sequences of leaves in allowed parse trees), but you also indicate

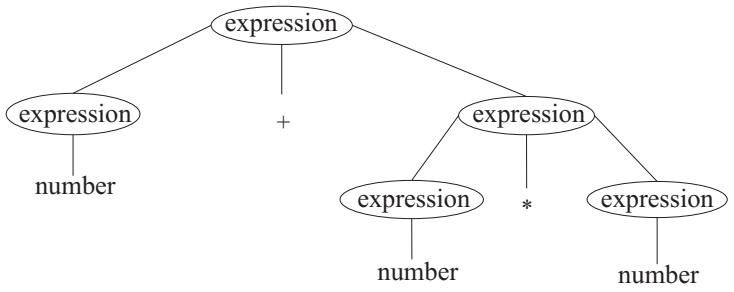


Figure 3.5: **A plausible tree structure for expression 4 + 9 * 2.** The tokens are leaves in the tree. Token **number** can have more than one lexeme, including 4, 9 and 2. This tree is only one way to describe expressions. The actual tree that you get depends on how you describe the syntax.

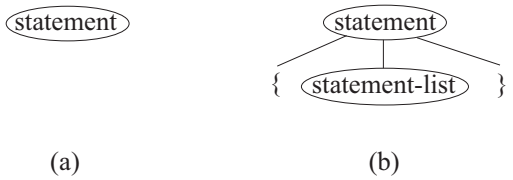
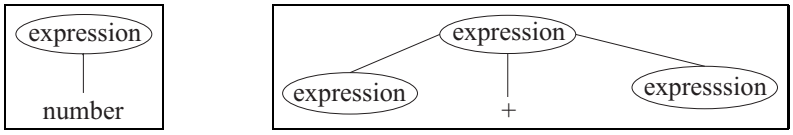


Figure 3.6: **Plausible tree structures for do-nothing statements in (a) Pascal and (b) C.** In Pascal, a statement can contain no tokens. In C, a statement-list can be empty, and a do-nothing statement can be written {}. When you read the tokens in a tree, you skip leaves that are labeled by nonterminals.



⟨expression⟩ ::= **number** ⟨expression⟩ ::= ⟨expression⟩ + ⟨expression⟩

(a) (b)

Figure 3.7: **Two productions.** Production (a) (⟨expression⟩ ::= **number**) indicates that a node labeled ⟨expression⟩ can have just one child, a leaf holding token **number**, as shown in the tree diagram above the production. Production (b) (⟨expression⟩ ::= ⟨expression⟩ + ⟨expression⟩) indicates that a node labeled ⟨expression⟩ can have three children, labeled as shown from left to right. The productions in a grammar tell all of the allowed local structures.

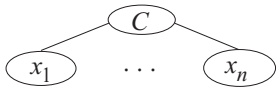
the structure of each program. We will use a notation called Backus-Naur Form (BNF) for describing parse trees.

Assume that the tokens of a language have already been described as part of the language's lexical description. To describe the language's syntax using BNF, you begin by defining a collection of nonterminals. When written outside of tree diagrams, nonterminals are usually written surrounded by angle brackets, to make it clear that they are nonterminals. For example, one of the basic units of a program is likely an expression, so you create a nonterminal ⟨expression⟩.

The next step is to describe what a parse tree can look like *locally* at a node that is labeled by a given nonterminal. That is done by writing a collection of *productions* for each nonterminal, where each production for nonterminal C gives one form that the tree structure just beneath a node labeled by C can look like. Figure 3.7 shows two examples of productions and corresponding local tree structures. In general, a production has the form

$$\langle \mathrm{C} \rangle ::= x_1 x_2 \ldots x_n$$

where ⟨C⟩ is a nonterminal, $n \geq 0$ and $x_1, \ldots, x_n$ are each either nonterminals or tokens, indicating that a parse tree can contain local structure



It is allowed for the right-hand side of a production to have no symbols, indicating that this nonterminal can be at a leaf in the parse tree. Only nonterminals that have such *erasing productions* are allowed to occur as leaves.

A language is described by a *context-free grammar*, with four parts: (1) a collection of nonterminals, (2) a collection of tokens, (3) a collection of productions telling all possible local tree structures, and (4) a nonterminal that is required to be the label of the root of a parse tree. Typically, only the collection of productions is shown, with the nonterminals and tokens being understood, and the root nonterminal understood to be the one on the

left-hand side of the first production. For example, the three productions

$$\begin{array}{lcl} \langle\text{expression}\rangle & ::= & \textbf{number} \\ \langle\text{expression}\rangle & ::= & \textbf{identifier} \\ \langle\text{expression}\rangle & ::= & \langle\text{expression}\rangle + \langle\text{expression}\rangle \end{array} \tag{3.2.1}$$

form a small grammar with just one nonterminal ⟨expression⟩, three tokens **number**, **identifier** and **+**, and with ⟨expression⟩ as the start nonterminal,

It is convenient to write several productions for the same nonterminal together, using a vertical bar in place of ::= in all but the first. For example, productions (3.2.1) can be written more succinctly as follows.

$$\begin{array}{lcl} \langle\text{expression}\rangle & ::= & \textbf{number} \\ & | & \textbf{identifier} \\ & | & \langle\text{expression}\rangle + \langle\text{expression}\rangle \end{array} \tag{3.2.2}$$

Keep in mind that a grammar indicates *all* of the local structure forms that you can use in a tree diagram. Every node labeled by a nontermal *must* have a structure immediately below it that is allowed by a production in the grammar.

3.2.4. Common grammatical forms

Choice

A statement might have one of several different forms. It might, for instance, be an assignment statement, an if-statement, a while-statement, a procedure call, or any of a large collection of different kinds of things. A grammar deals with that by including a production for each form. An example of productions for statement might include the following.

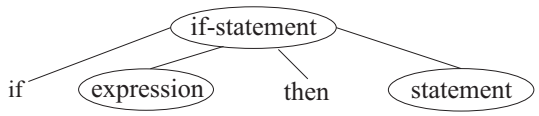
$$\begin{array}{lcl} \langle\text{statement}\rangle & ::= & \langle\text{assignment-statement}\rangle \\ & | & \langle\text{if-statement}\rangle \\ & | & \langle\text{while-statement}\rangle \\ & | & \langle\text{procedure-call}\rangle \end{array} \tag{3.2.3}$$

Recursion

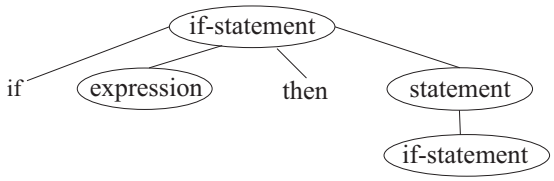
A statement often contains other statements embedded within it. For example, a Pascal if-statement has either the form **if** *expression* **then** *statement* **else** *statement* or the form **if** *expression* **then** *statement* (where there is no else-part). The following productions describe the form of a Pascal if-statement.

$$\begin{array}{lcl} \langle\text{if-statement}\rangle & ::= & \textbf{if}\ \langle\text{expression}\rangle\ \textbf{then}\ \langle\text{statement}\rangle\ \textbf{else}\ \langle\text{statement}\rangle \\ & | & \textbf{if}\ \langle\text{expression}\rangle\ \textbf{then}\ \langle\text{statement}\rangle \end{array} \tag{3.2.4}$$

From a parser tree perspective, the second of those production says that a tree can contain the following local structure.



The productions clearly show that an if-statement has a smaller statement as part of it. That statement might be another if-statement, yieiding the following structure.



Recursion, where a nonterminal occurs in a subtree of another node labeled by the same nonterminal, is fundamental to writing grammars.

Repetition

Often you find yourself wanting to describe a sequence of similar things. For example, in the programming language Lisp, one form of expression consists of a sequence of zero or more Lisp expressions, written one after another, enclosed in parentheses. So one production for Lisp expressions should be

$$\langle \text{lisp-expression} \rangle ::= \textbf{(}\ \langle \text{lisp-expression-list} \rangle\ \textbf{)} \tag{3.2.5}$$

(Note the parentheses in the production. They are tokens. There are three things on the right-hand side.) Nonterminal ⟨lisp-expression-list⟩ describes sequences of expressions. Each sequence can either be the empty sequence, containing no expressions, or consists of an expression followed by (zero or) more expressions. So productions for ⟨lisp-expression-list⟩ are as follows.

$$\begin{array}{lll} \langle \text{lisp-expression-list} \rangle & ::= & \\ & | & \langle \text{lisp-expression} \rangle\ \langle \text{lisp-expression-list} \rangle \end{array} \tag{3.2.6}$$

The right-hand side of the first production is empty.

A procedure call in a particular language might allow a list of expressions, separated by commas, as a parameter list. A nonempty parameter list is either a single expression or a single expression followed by a comma followed by more expressions. A grammar for ⟨procedure-call⟩ and ⟨expression-list⟩ goes as follows.

$$\begin{array}{lll} \langle \text{procedure-call} \rangle & ::= & \textbf{identifier}\ \textbf{(}\ \langle \text{expression-list} \rangle\ \textbf{)} \\ \langle \text{expression-list} \rangle & ::= & \langle \text{expression} \rangle \\ & | & \langle \text{expression} \rangle\ \textbf{,}\ \langle \text{expression-list} \rangle \end{array}$$

3.2.5. Ambiguity

The following simple grammar describes expressions with operators +, − and * as well as parentheses.

$$\begin{array}{lll} \langle \text{expression} \rangle & ::= & \textbf{number} \\ & | & \langle \text{expression} \rangle + \langle \text{expression} \rangle \\ & | & \langle \text{expression} \rangle - \langle \text{expression} \rangle \\ & | & \langle \text{expression} \rangle * \langle \text{expression} \rangle \\ & | & \textbf{(}\ \langle \text{expression} \rangle\ \textbf{)} \end{array} \tag{3.2.7}$$

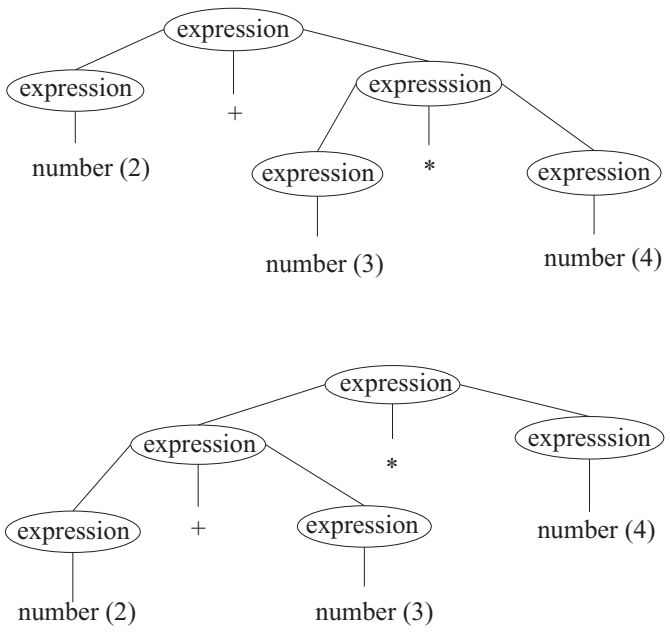


Figure 3.8: **Two parse trees for $2 + 3 * 4$ according to Grammar (3.2.7).** Since there are two different parse trees for the same sequence of tokens, Grammar (3.2.7) is ambiguous.

The syntax of a language is supposed to indicate something about the structure of programs, and, although the syntax does not tell us that + means addition and * means multiplication, it should tell us whether the structure of expression 2+3∗4 is more like 2+(3∗4) or (2+3)∗4. Unfortunately, Grammar (3.2.7) does not do so. Figure 3.8 shows two different parse trees for expression $2 + 3 * 4$, both allowed by the grammar, but showing different structure. We say that grammar G is *ambiguous* if it is possible to produce two different parse trees, both using the rules of G, that have exactly the same sequence of tokens in their leaves. Figure 3.8 demonstrates that Grammar (3.2.7) is an ambiguous grammar.

Fortunately, it is possible to create an unambiguous grammar for simple expressions. A way to overcome ambiguity is to add more nonterminals to the grammar, each representing a level of precedence. It is also important to avoid symmetries that lead to ambiguity. Grammar (3.2.8) does both.

$$
\begin{array}{rcl}
\langle \text{expression} \rangle & ::= & \langle \text{term} \rangle \\
& | & \langle \text{expression} \rangle \ + \ \langle \text{term} \rangle \\
& | & \langle \text{expression} \rangle \ - \ \langle \text{term} \rangle \\
\langle \text{term} \rangle & ::= & \langle \text{factor} \rangle \\
& | & \langle \text{term} \rangle \ \textbf{*} \ \langle \text{factor} \rangle \\
\langle \text{factor} \rangle & ::= & \textbf{number} \\
& | & \textbf{(} \ \langle \text{expression} \rangle \ \textbf{)}
\end{array}
\tag{3.2.8}
$$

The idea is that ⟨expression⟩ can produce any kind of expression, ⟨term⟩ only allows operator + to occur inside parentheses (but allows * to occur outside parentheses), and ⟨factor⟩ does

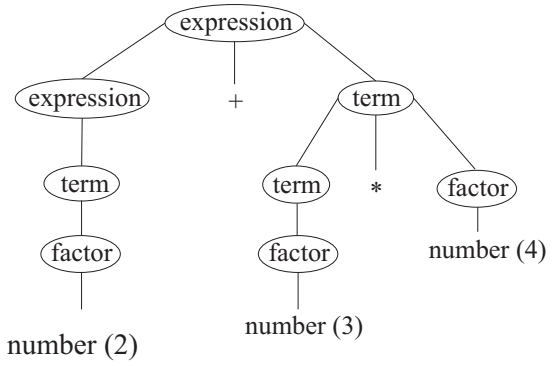


Figure 3.9: **The unique parse tree for expression** $2+3*4$ **according to Grammar (3.2.8).**

not allow any operators to occur outside of parentheses. Figure 3.9 shows a parse tree for $2 + 3 * 4$ using Grammar (3.2.8). Try to convince yourself that Grammar (3.2.8) describes the same expressions as Grammar (3.2.7), but that it is unambiguous. Why aren't there other ways to derive $2 + 3 * 4$?

Associativity

Operator precedence tells how expressions involving operators of different precedence levels are parsed. But that leaves the issue of how to parse expressions involving more than one occurrence of the same operator, or of operators with the same precedence. For example, should $5 - 2 - 2$ be thought of as similar to $5 - (2 - 2)$ or to $(5 - 2) - 2$? That is where rules of *associativity* come into play. An operator is left-associative if several occurrences of it are done from left to right, and it is right-associative if operations are done from right to left. If subtraction is left-associative, then $5 - 2 - 2$ is understood to have a structure similar to $(5 - 2) - 2$.

Grammar (3.2.8) already says that operators +, − and * are left-associative. Look at the following two productions. The first is taken from Grammar (3.2.8), and the second is a modification of that production.

$$\begin{aligned} \langle \text{expression} \rangle &::= \langle \text{expression} \rangle + \langle \text{term} \rangle \\ \langle \text{expression} \rangle &::= \langle \text{term} \rangle + \langle \text{expression} \rangle \end{aligned}$$

According to the first of those two productions, expression $2 + 3 + 4$ should be taken to be an expression followed by + followed by a term. Clearly, the expression is $2+3$ and the term is 4 (which is just a factor), since a term does not allow + to occur outside parentheses. So that production indicates that + is left-associative. The second production, on the other hand, indicates that $2 + 3 + 4$ should be a term followed by + followed by an expression. In that case, the term must be 2 and the expression $3 + 4$, so that addition is right-associative. You should include only one of those two productions in a grammar.

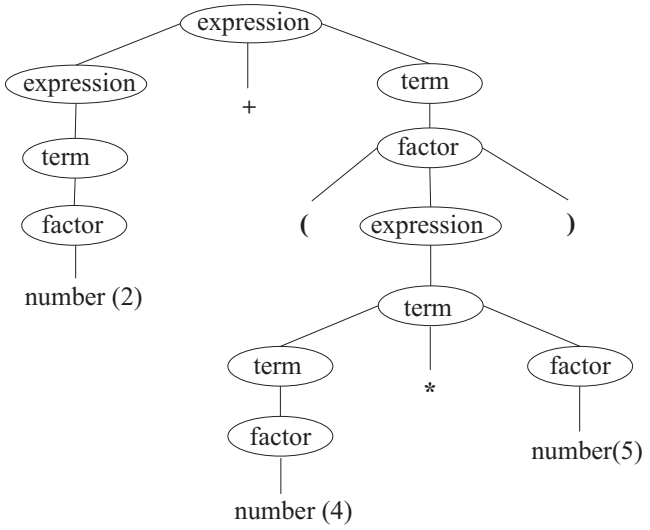


Figure 3.10: **A parse tree for expression 2 + (4 * 5), using Grammar (3.2.8).**

3.2.6. Syntax is not meaning

When describing a language using BNF, be careful to remember that you are concerned with the syntax of the language, not so much with meaning. Grammar (3.2.8) allows expressions that include parentheses. If you think about what the expression means, the parentheses do not indicate that any operations are to be performed. But from a syntactic point of view, parentheses are certainly tokens, and are part of any program that uses them. Figure 3.10 shows a parse tree for $2 + (4 * 5)$ using Grammar (3.2.8). The parentheses are leaves, since they are tokens that belong to the expression.

Does it matter whether addition is performed from left to right or from right to left? Since addition is an associative operator, you might argue that it does not. But from a syntactic perspective, it definitely does matter. In order to be unambiguous, a grammar needs to ensure that no sequence of tokens can be parsed two different ways, even if those two ways would have the same meaning.

3.2.7. Extended BNF notation

Some simple extensions make BNF notation a little more convenient to use. To make it clear that we are not using ordinary BNF, any grammar written with the extensions is said to be an Extended BNF (EBNF) grammar. Placing a sequence of tokens and nonterminals in square brackets indicates that they are optional. For example, production ⟨expression⟩ ::= ⟨term⟩ [+ ⟨expression⟩] is a shorthand for the two productions

$$\begin{array}{lcl} \langle \text{expression} \rangle & ::= & \langle \text{term} \rangle \\ & | & \langle \text{term} \rangle + \langle \text{expression} \rangle \end{array}$$

Placing a sequence of tokens and nonterminals in curly braces indicates zero or more repetitions of that sequence. For example, production $\langle A \rangle ::= \langle B \rangle \ \{\langle C \rangle\}$ indicates that $\langle A \rangle$ can

be ⟨*B*⟩ or ⟨*B*⟩⟨*C*⟩ or ⟨*B*⟩⟨*C*⟩⟨*C*⟩, etc., with as many copies of ⟨*C*⟩ as desired.

You never really need brackets or braces. They are only conveniences. For example, you can derive the same sequences of tokens as are described by production ⟨*A*⟩ ::= ⟨*B*⟩ {⟨*C*⟩} from the following productions.

$$\begin{array}{rcl} \langle A \rangle & ::= & \langle B \rangle \langle \text{rest} \rangle \\ \langle \text{rest} \rangle & ::= & \\ & | & \langle C \rangle \langle \text{rest} \rangle \end{array}$$

3.2.7.1. Examples using EBNF

Grammar (3.2.8) can be written as follows using EBNF.

$$\begin{array}{rcl} \langle \text{expression} \rangle & ::= & [\langle \text{expression} \rangle \ \mathbf{+} \ ] \ \langle \text{term} \rangle \\ & ::= & [\langle \text{expression} \rangle \ \mathbf{-} \ ] \ \langle \text{term} \rangle \\ \langle \text{term} \rangle & ::= & [\langle \text{term} \rangle \ \mathbf{*} \ ] \ \langle \text{factor} \rangle \\ \langle \text{factor} \rangle & ::= & \mathbf{number} \\ & | & \mathbf{(} \ \langle \text{expression} \rangle \ \mathbf{)} \end{array}$$

In some places, repetition can be used in place of recursion. For example, the rule for ⟨term⟩ can be replaced by the following extended production, but doing so no longer allows you to express associativity, so you need to add, on the side, that * is left-associative.

$$\langle \text{term} \rangle \ ::= \ \langle \text{factor} \rangle \ \{\mathbf{*} \ \langle \text{factor} \rangle\}$$

Grammar (3.2.4) uses two rules to show the possible forms of a Pascal if-statement. With EBNF, you can do the same in a single production.

⟨if-statement⟩ ::= **if** ⟨expression⟩ **then** ⟨statement⟩ [**else** ⟨statement⟩]

A Pascal compound statement starts with the token **begin** and ends with the token **end**. In between them is a sequence of zero or more statements, separated by semicolons. A single EBNF rule suffices to describe that.

⟨compound-statement⟩ ::= **begin** [⟨statement⟩ {; ⟨statement⟩}] **end**

3.2.8. Syntax diagrams

Syntax diagrams are a graphical form of grammar. Figure 3.11 shows a diagram of simple expressions. The nonterminals are shown in rectangles and the tokens are shown in ovals or circles.

The idea is that, to derive an expression, you follow the arrows in the diagram for expression, writing things encountered along the way. At a token node, you write the token. At a nonterminal node, you must write any sequence of tokens that can be derived according to the diagram for that nonterminal. When there is a fork in the arrows, you have a choice of either branch. To complete an expression, you must reach the end of the diagram for expressions (the arrow at the right side).

One advantage of syntax diagrams is that it is possible to show loops. Figure 3.12 shows an alternative diagram for expressions. Their graphical nature makes syntax diagrams a little easier to read than BNF notation, so they are a popular way of describing syntax. Unfortunately, syntax diagrams with loops suffer from the same problem as EBNF that associativity information is not present in them, and needs to be indicated by remarks that are not part of the grammar itself.

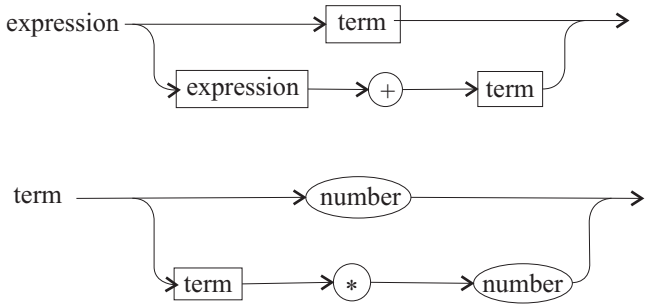


Figure 3.11: **A pair of syntax diagram that describe expressions and terms, with operators + and *.** Nonterminals are shown in rectangular boxes and tokens are shown in circular or oval boxes. In this grammar, a term directly produces a number; there is no factor nonterminal.

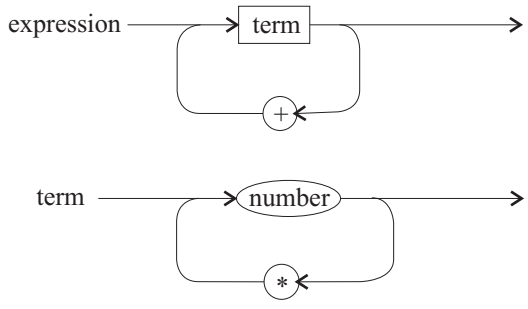


Figure 3.12: **Another pair of syntax diagrams that describe expressions and terms.** In these diagrams recursion has been replaced by cycles in the diagrams.

3.3. Exercises

3.1. C++ was developed by augmenting C. A C comment begins with /* and ends with */. When C++ was developed, it was deemed necessary to add another kind of comment that begins with // and ends at the end of a line. Explain why the C style of comments was considered so bad that it needed to be replaced. Also explain how a compiler can warn a programmer about some mistakes using the C comment.

3.2. How could the designers of Fortran have avoided the problem shown in Figure 3.3 without changing the syntax of a do-statement?

3.3. Give arguments for and against the use of layout syntax. (You might find Exercise 3.7 helpful.)

3.4. Take a position either that the C choice of using / for integer division or the Ada choice of the word **div** is preferable. Offer justification for your choice.

3.5. In C, an assignment $x = y$ (copy the value of y into variable x) is an expression whose value is the value of y. Expression $x == y$ is true if x and y are equal. But in C, the integer 0 is used to indicate a false result and 1 is used to indicate true. (There is no *boolean* type.) So a condition, such as $x == y$, is considered to be an integer. Suppose that x and y are integers. Explain what happens if you write a statement that begins `if(`x `=` y`)` ..., accidentally writing = instead of ==. Can you suggest ways in which the language designers could have avoided this problem?

3.6. The syntax of C and Java each allows a statement to be just a semicolon, indicating that nothing should be done. Give an example where an extraneous semicolon (easily overlooked by a programmer) vastly changes the meaning of a C or Java program fragment. (Hint: look at a while loop.)

3.7. The body of a C while loop is one statement. Suppose a program contains

```
while(f(x) > 0)
  x = x + 1;
```

Now you decide to print the value of x each time around the loop, writing

```
while(f(x) > 0)
  x = x + 1;
  print(x);
```

Explain why this does not have the desired effect. Would layout syntax help to avoid this problem? Can you suggest a way to avoid it, or at least to reduce its likelihood, in a free-form language?

3.8. What is the difference between a lexeme and a token? Why is it useful to make that distinction?

3.9. Does every token have exactly one corresponding lexeme? Explain.

3.10. Should a string constant such as “the daisy” be considered a single lexeme, even though it contains a space?

3.11. What is the typical rule that is used to break a sequence of characters up into a sequence of lexemes? How do you know where each lexeme begins and ends?

3.12. If a character is not part of a lexeme, can it always be deleted without changing the meaning of a program?

3.13. Using Grammar (3.2.8) draw a parse tree, starting at ⟨expression⟩, for each of the following expressions.

(a) $3 + 4 + 8$

(b) $((5))$

(c) $18 * 2 + 99$

(d) $18 + 2 * 99$

(e) $(4) + 5 * (6 + 2)$

3.14. According to Grammar (3.2.8), is multiplication left-associative or right-associative?

3.15. Show a parse tree for string *aacacab* according to the following grammar, where the start symbol is ⟨S⟩, and the tokens are a, b and c.

$$\begin{array}{rcl} \langle S \rangle & ::= & \langle F \rangle\ a\ \langle S \rangle \\ & | & b \\ \langle F \rangle & ::= & a\ \langle F \rangle \\ & | & c \end{array}$$

3.16. Combine Grammars (3.2.5) and (3.2.6), and add one more production,

$$\langle \text{lisp-expression} \rangle\ ::=\ \text{x}$$

where x is a token. Using that grammar, show a parse tree for each of the following, starting with ⟨lisp-expression⟩.

(a) ( )

(b) (x x)

(c) ((x) x)

(d) ((x x x))

3.17. Show that Grammar (3.2.4) is ambiguous. Assume that there is a simple kind of statement s and a simple kind of expression e, and do not worry about the details of those.

3.18. Show that the following grammar is ambiguous. The start symbol is ⟨S⟩, and a is a token.

$$\begin{array}{rcl} \langle S \rangle & ::= & \langle S \rangle\ a \\ & | & a\ \langle S \rangle \\ & | & a \end{array}$$

3.19. Explain why Grammar (3.2.8) is unambiguous. Do not merely repeat what it means for it to be unambiguous, but argue that it meets the requirements of an unambiguous grammar.

3.20. Lexical rules need to specify collections of lexemes that correspond to a particular token. Sometimes grammars are used for that purpose. A Pascal identifier consists of a letter followed by zero or more letters and digits. Assume that nonterminals ⟨letter⟩ (a single letter) and ⟨digit⟩ (a single digit) are available. Give a BNF grammar for ⟨identifier⟩, describing Pascal identifiers. Do not use extensions.

3.21. Show how EBNF can be used to simplify the grammar from Exercise 3.20.

3.22. Write BNF productions for ⟨while-statement⟩ in a grammar for C or Java.

3.23. Write a BNF grammar that derives all (and only) sequences of the symbols `(`, `)`, `[` and `]` that are well-nested. In a well-nested sequence, parentheses must be balanced and brackets must be balanced, and balanced parts cannot partially overlap. For example, `[](([][()])(())` is a well-nested sequence, but `([)]` and `))((` are not well-nested.

3.24. Write a BNF grammar that derives all (and only) sequences of the letters a, b and c where all of the a's precede all of the c's.

3.25. Write a BNF grammar that derives all (and only) sequences of the letters a and b that have the same number of a's as b's. (This one will take some thought.)

3.26. Draw a syntax diagram or diagrams that describe Pascal identifiers. See Exercise 3.20.

3.27. Draw a syntax diagram that describes a Pascal compound statement. You can use nonterminal ⟨statement⟩.

3.28. Draw a syntax diagram that describes a Pascal if-statement. You can use nonterminal ⟨statement⟩.

3.29. Draw a syntax diagram that describes a C or Java if-statement. Allow a version with an else part, and also a version without an else part.

3.30. A while loop involves repetition. Does that mean you need to show a loop in a syntax diagram for a while statement? Draw a syntax diagram for a C or Java while-loop.

3.4. Bibliographic notes

Van der Linden [64] describes the NASA Fortran bug in more detail. Algol 58 is described by Backus [9], and BNF is introduced in the Algol 60 report [75]. The Pascal report [57] uses syntax diagrams to describe the syntax of Pascal. Chomsky independently discovered grammars, introducing them in [24] and [25]. Grammars and lexical rules are both used heavily in building compilers, and books on compilers ([5], [48]) provide extensive material on both.

Chapter 4

Data and Data Representation

4.1. Programs and data

A program works by manipulating information. It reads information, stores it in memory, and produces information as its answers. Information, or *data*, comes in various forms; characters, integers, strings, lists and trees are just a few. Anything that might be stored in a variable, passed to a subprogram, returned from a subprogram, or stored in the computer's memory is a form of data.

Data can be viewed at two levels of detail. There is a conceptual level, where you think of the data in your own terms. For example, you understand integers quite well, since you have used them since childhood. There is also a more concrete view, where you see the data in terms of how it is stored in a computer's memory. The integer 2 might be represented by a sequence of bits, such as 00000010. The conceptual view is the *semantics* of the data, while the representation is a way of achieving the semantics. This chapter explores both views of data.

4.2. Simple values

When a novice programmer is first introduced to a programming language, usually the first kinds of data that he or she sees are numbers. Numbers are well understood from the early years of school, so they provide a familiar setting in which to study the concepts of computation. Fortunately, numbers are also among the most useful of data forms, and provide some excellent examples of computational problems. Numbers are examples of *simple values*, which are any values that are *atomic* because they have no discernible internal structure, and are not intended to be taken apart into smaller pieces.

Other simple values include truth values (true and false), also called *boolean* values, used as the results of tests; and characters, used to store textual information. Some languages provide the concept of a *symbol*, which is a thing that is described entirely by its name. For example, **robot** and **vegetable** are symbols. A symbol has no structure — it is just a thing. It is left up to the programmer to interpret what that thing means to him or her.

Simple values are typically stored in fixed size chunks of memory. For example, a

character is stored as a pattern of bits.[1] For a truth value (true or false) you could make use of a single bit, but you typically use one byte simply because smaller pieces of memory are difficult to use in most computers. A symbol is usually represented as the address of a piece of memory where the name and possibly other information about the symbol are stored.

Integers are also typically stored in fixed size chunks of memory. On a 32-bit computer, an integer is normally stored as a 32-bit binary number in a single four-byte word, which is convenient since the processor handles instructions that manipulate single-word integers. But using 32 bits has the disadvantage that the largest integer that can be stored is $2^{31} - 1$ (about two billion). Since the available representations encode both positive and negative integers, adding 1 to the largest positive integer yields a negative integer.

How a programming language implementation chooses to store integers is a reflection of what the language is trying to provide to the programmer. The language C, for example, attempts to provide a convenient but relatively direct view of the computer hardware. Integers are stored in one word, and when you see two integers added together in a C program, you understand that the addition is to be done by a single machine ADD instruction. But some languages want to provide mathematically correct integers, where adding 1 to a positive integer always yields a larger positive integer. In that case, an integer needs to be stored in several bytes, and arithmetic operations are carried out by software rather than hardware, at a cost in efficiency. Mathematica , which is designed for general mathematical calculations and reasoning, is an example of a programming language that offers this more conceptual, and mathematically correct, view of integers.

Real numbers, such as 2.4, are typically stored in 64 bit words as binary numbers with limited precision, since processors can perform arithmetic on such numbers. Each real number is stored in *floating point* representation, as a pair of integers called the *mantissa* and the *exponent*, where a number with mantissa 110111001110_2 and exponent e represents the real number $0.110111001110_2 \times 2^e$. Of course, for 64 bits numbers, the mantissa would be longer. The IEEE 64-bit standard uses 11 bits for the exponent and 53 bits for the mantissa (including a sign bit in each). That allows positive numbers as small as 10^{-323} and as large as 10^{308}, stored with about 15 decimal digits of precision, as well as the corresponding negative numbers. A number such as 1/3 cannot be written exactly in binary notation, so it is stored as an approximation. Even the number 0.1, which can be written exactly in decimal notation, can only be written approximately in binary notation. (It is $0.000110011001100\ldots$ in base 2.)

Since Mathematica wants to present a mathematical view of numbers, it stores real numbers with arbitrary precision, using as many bits as you ask for. In fact, Mathematica can even use a symbolic notation to represent irrational numbers such as π exactly, by simply calling numbers by their names, or by saying how they would be computed to arbitrarily high precision.

4.3. Tuples and records

A value that has visible internal structure is called a *complex value*. One kind of complex values is an ordered pair, which should be familiar from mathematics as a point in the Cartesian plane. For example, the point with x-coordinate 3 and y-coordinate 5 is written

[1] ASCII uses one byte (7 or 8 bits), Unicode uses two bytes, and extensions of Unicode use up to 4 bytes per character.

$(3, 5)$. The concept of an ordered pair can be extended to an ordered *tuple*, such as $(3, 5, 7)$, consisting of two or more values. One of the most common uses of tuples is to pass information to subprograms; a subprogram that takes three parameters can be viewed as taking a tuple of three things.

A key feature of a tuple is that it has a known size. If you create a function that takes four parameters, you know, while writing the function definition, that there must be exactly four parameters, so the parameter is a tuple of size four. When a program asks for a part of a tuple, it explicitly specifies which one it wants, such as the third one. It never asks for the k-th component, where k is a variable. A consequence of that for a typed language, where the compiler performs type checking, is that the components of a tuple do not all need to have the same type. A compiler would not be able to determine the type of the k-th component if it does not know, when compiling the program, what k is.

A *record* is a kind of tuple where the parts are identified by names instead of by positions. For example, tuple (true, "kangaroo", 50), whose components are identified by positions 1, 2 and 3, is equivalent to the record

Key	Information
first	true
second	"kangaroo"
third	50

whose components are called first, second and third. Of course, you would normally choose much more descriptive names. Programmers have a much easier time remembering the names of components (such as **social** for a social security number) than remembering to look in the third position of a tuple. A compiler treats the two in the same way, remembering the position corresponding to each name. Ada and C are examples of languages that provide records; Ada calls them records, while C calls them structures.

Representing tuples

An efficient way to store a tuple is to put the parts in consecutive memory locations. An ordered pair consisting of a four-byte value and an eight-byte value would occupy 12 consecutive bytes. Since the members of the tuple are stored sequentially in memory, this is called *sequential representation* of the tuple.

4.4. Lists and arrays

A tuple is a kind of sequence of a fixed size. A *list* (sometimes called an *array*) is another kind of sequence whose size, or *length*, can only be determined by examining it; a variable that holds a list could hold a list of any length. A typical operation is to select the k-th member of a list, where k is a variable or other expression. In a typed language, where the compiler needs to know the type of each thing, all members of a list must have the same type, since otherwise it is impossible to know the type of the k-th member. To avoid confusion between the two closely related notions of tuple and list, we write lists inside square brackets. For example, $[2, 3]$ is a list of two numbers, but $(2, 3)$ is a tuple of two numbers.

A list can have any length, including zero; [ ] indicates an empty list. The *singleton* list [2] has only one member, making it tempting to say that it *is* exactly the same as the number 2. But in programming languages, it is usually necessary to keep lists and numbers separate. A list of one thing is not the same thing as its first (and only) member.

You can mix complex values in arbitrary ways. For example, [(2,5), (4,3), (9,7)] is a list of ordered pairs whose first member is the pair $(2, 5)$. Similarly, [[2,3], [ ], [4,8,10], [0]] is a list of lists and ([5,4], [6]) is a tuple of two lists.

Sequential representation of lists

Like a tuple, a list can be stored in memory by putting its members one after another. But unlike tuples, lists require additional length information that needs to be stored with the list. One convention is to store the length at the start of the representation, storing list ['a', 'b', 'c'] in the same way as tuple (3, 'a', 'b', 'c'). That makes it fast and convenient for a program to find the length information. A *string* is a list of characters, and some sequential implementations of strings store the length of the string in the first byte or two bytes.

An alternative way to indicate the length is to put some kind of special marker just after the list in memory. That requires a marker value to be available to use. For example, strings typically do not contain the null character (whose character code is 0, and which shows up as nothing at all when printed), so the null character is available as an end marker. To store list ['a', 'b', 'c'] with an end marker, you represent the list as the tuple ('a', 'b', 'c', **null**).

Sequential representation has some desirable characteristics. It is easy to implement, since a computer's memory is already set up as a sequence of bytes, and the computer hardware already performs the basic operations needed to manage sequential lists. A major advantage is that accessing the k-th value in the list, for a given k, is very efficient. If the list begins at memory address p, and each item in the list occupies b bytes, then the k-th value in the list begins at memory address $p + b(k - 1)$. A little arithmetic gets the address of the k-th value, in an amount of time that is the same regardless of how large k is. This *random access* property can be critical to the efficiency of some programs.

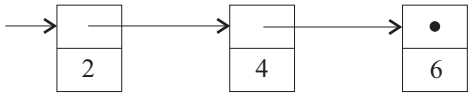
Unfortunately, the sequential representation method suffers from some difficulties. Suppose that you have a list $[40, 50, 60]$ of three numbers, represented as tuple $(3, 40, 50, 60)$, and you want to build the list $[30, 40, 50, 60]$ of four values, by adding 30 to the front of what you already have. To build the representation of the new list, you must find an unoccupied place in memory that is large enough to hold the new list and copy the old list into that memory, producing, sequentially, $(4, 30, 40, 50, 60)$. The amount of time and memory required to add a value to the front or back of a list using sequential representation is proportional to the length of the list.

Linked representation of lists

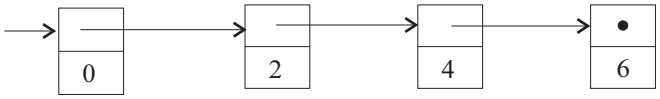
Think again about a computer's memory. Each byte in the memory has an *address*, an integer telling where that byte is located in the memory, and a program can refer to any byte by its address. It can refer to a section of several consecutive bytes by referring to the address of the first byte in the section. But memory addresses are just integers, and so they can be stored in the memory. That allows one data item to refer to another data item by indicating the address where that other item is stored. When a variable or data item

contains the address of another variable or data item, the address is usually called a *pointer* or a *reference*.

There is a way of storing lists that can avoid some of the copying that is required by the sequential representation. *Linked representation* stores the list $[2, 4, 6]$ in linked *cells*, each a small block of memory holding one thing and a pointer to the next cell. The last cell holds the *null pointer* 0, indicating the end of the list. (Memory address 0 is generally not an allowed memory address, so is available as a special marker.) In the following diagram of $[2, 4, 6]$ in linked form, the null pointer is shown as a bullet.



To add 0 to the beginning of this list is easy. Just create a new cell holding 0 and a pointer to the cell that is at the start of the previous list. The new list is as follows.



You can view the addition of 0 to the front as a change to the list. But notice that the representation of list $[2, 4, 6]$ is still intact, beginning with the cell that contains 2. So both lists $[2, 4, 6]$ and $[0, 2, 4, 6]$ are present. That allows you to think in terms of creating a new list without destroying the old one. The two lists share memory, which makes it possible to add a value to the front of a list in a constant amount of time, regardless of how long the list is, without destroying the list that was extended. Memory sharing also has important ramifications for programs that change things. Any change to the representation of list $[2, 4, 6]$ will implicitly change the representation of list $[0, 2, 4, 6]$ as well. Either no changes should be allowed, or the programmer should be aware that changes to one list will affect the other.

The linked representation makes use of an important characteristic of nonempty lists. The first cell of a nonempty list contains just two pieces of information: the first member of the list and a pointer to the rest of the list. So the information contained in a nonempty list can always be expressed as an ordered pair consisting of the first member of the list and the list of all members excluding the first member. For example, list $[2, 4, 6]$ can be broken into the ordered pair (2, [4, 6]). If we already have a way to represent ordered pairs (such as sequential representation), and we throw in a way to represent empty lists (such as using a null pointer) then we can represent arbitrary lists. That is exactly what the linked representation does. The first member of a nonempty list is called the *head* of the list, and the list of all remaining members is called the *tail* of the list.

The linked scheme has some advantages over the sequential representation scheme, since it allows lists to be extended at their front cheaply. But it suffers disadvantages as well. An obvious one is that it uses more memory. If the members of the list use as much memory as pointers, then about half of the memory is used by the pointers, and the linked representation scheme uses about twice as much memory as the sequential representation scheme to store the same list.

Locating the k-th member of a linked list is much slower than locating the k-th member of a sequentially stored list, since you need to chain through all of the previous members, which takes time proportional to k. That compares to a constant amount of time, independent of k, for the sequential representation.

Choosing a representation

The representation that a language implementation chooses depends on the operations that it encourages programmers to use. C encourages programmers to use indexing (selecting the k-th member), so it represents arrays using sequential representation. Lisp encourages programmers to use a function called cons that adds a new value to the beginning of an existing list; for List, it makes more sense to use linked representation.

Some language implementations use other representations of lists. The implementation of Cinnameg uses a hybrid representation that combines linked and sequential representation with another representation that uses trees as representations of lists (see the Section 4.6). The hybrid representation makes it possible to concatenate lists in a constant amount of time, something neither the linked nor sequential representations of lists can do. But the scheme requires the use of memory sharing in ways that are difficult to understand. As noted earlier, the programmer must either be aware of all memory sharing or must not be able to make modifications to data structures. Since the programmer cannot be made aware of the memory sharing in its lists, Cinnameg does not allow the programmer to rearrange a list once it is built.

4.5. Sets

Lists are ordered, and allow duplicates. For example, list $[8, 2, 8, 4]$ contains two occurrences of 8, and they occur in the first and third positions. *Sets*, on the other hand, are unordered and do not allow duplicates. Set $\{2, 4, 6\}$ is the same as set $\{6, 4, 2, 2\}$. SETL and Griffin are examples of languages that use sets as a fundamental data structure. Operations include intersection, union, creating a singleton set, etc.

One advantage of using sets is that there are so many choices that can be made for representation. You can use the list $[2, 4, 6]$ as a representation of the set $\{2, 4, 6\}$, but other representation schemes include binary search trees, hash tables and, for sets of small integers, bit arrays, where A[k] is 1 when k is a member of the set and 0 when k is not a member. The range of choices makes room for a powerful compiler to make intelligent choices that speed up program execution.

4.6. Trees

One frequently used kind of value is a *tree*, such as the one shown in Figure 4.1(a) A tree has a *root value* and zero or more *subtrees*, which are ordered from left to right. The tree shown in Figure 4.1(a) has a root value of 4, and has three subtrees, shown in Figure 4.1(b). A tree with three subtrees can be represented as an ordered pair (root-label, [subtree-1, subtree-2, subtree-3]). As long as we know how to represent ordered pairs and lists, we know how to represent trees.

Using trees to represent other kinds of data

Trees are amazingly versatile in what they can do. You can view ordered pairs and lists as special kinds of trees if you like. Recall that a *symbol* is a kind of simple value that is described by its name. A convenient way to represent an ordered pair as a tree is to use

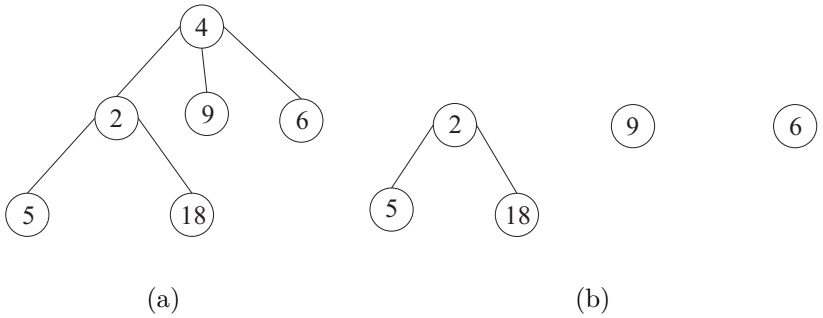
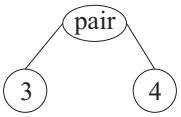
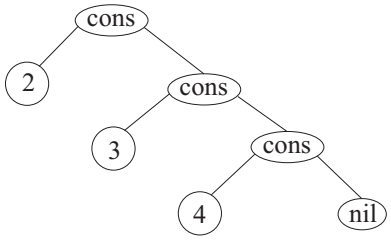


Figure 4.1: **(a) An example of a tree data structure and (b) the three subtrees of tree (a).**

symbol **pair** as the root label, with the two parts of the pair represented by two subtrees. For example, ordered pair $(3, 4)$ might be represented by the tree following tree.



Lists can also be represented as trees in several different ways. One popular way is to assume that there are symbols *cons* and *nil* and to represent list $[2, 3, 4]$ as follows.



A tree with just one node that holds symbol nil stands for the empty list. Notice that it is easy to find the head (in the left subtree of the root) and the tail (in the right subtree of the root). In fact, the tree representation is really just the linked representation turned diagonally.

There are other ways to use trees to represent lists. For example, a tree might, by convention, be traversed in preorder to find the members of the list. (A preorder traversal first selects the root, then does preorder traversals of the subtrees, in left-to-right order.) List $[3, 6, 4, 7, 9, 5]$ could be represented as the tree in Figure 4.2. That probably sounds like a strange choice, but it has the advantage that there are many different ways to represent a list, and that flexibility can be exploited to implement some operations on lists very efficiently. It forms the basis of one of the ways in which lists are represented in the Cinnameg implementation.

You might wonder why lists and tuples would be represented as trees when earlier we did the opposite, representing trees in terms of tuples and lists. But what happens inside a

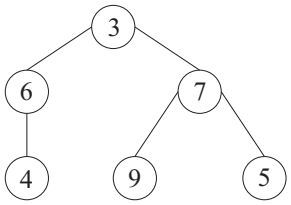


Figure 4.2: **A tree that can be used as a representation of list [3, 6, 4, 7, 9, 5].** The tree is traversed in preorder, where you start with the root label, then traverse the left subtree in preorder, then traverse the right subtree in preorder. The head of the list is easy to get because it is stored at the root of the tree.

Key	Information
name	"Minerva R. Forester"
occupation	"Forest Ranger"
social	"111-11-1111"

Figure 4.3: **A diagram of a table, or dictionary.** A table consists of a collection of rows, where each row has a key and some information associated with that key. No two rows contain the same key.

language implementation is invisible to the programmer. The point is that a language that only supplies trees does not really need to supply lists and pairs separately. Prolog is an example of a language that uses trees, and expresses other data structures in terms of them.

Trees are written in programming languages in a variety of ways. Prolog uses notation pair(3,4) for the tree with root label *pair* and two subtrees, 3 and 4. In Cinnameg, you write r-<L to describe a tree whose root label is r and whose subtrees are the trees in list L.

4.7. Tables

A *table* is a data item that you can think of as a collection of rows, each having two columns called the *key* and the *information*. No two rows have the same key value. Figure 4.3 shows an example of a table. The keys can, in principle, be any kind of values, not just strings or names, but keys that are strings are very common. Operations on tables include lookups (finding a row with a particular key and returning the associated information) adding a new row and removing the row that has a given key. Python and Cinnameg are examples of languages that provide tables as fundamental data types. (Python calls tables *dictionaries*.)

A table is a set of rows. So any representation of sets that allows the members of the sets to be ordered pairs can be used to represent tables. Typical representations are binary search trees and hash tables.

4.8. First class data items

Numbers are examples of data items that a program can store and manipulate. You can pass a number to a subprogram, return a number from a function, put a number into a variable, etc. Depending on the programming language, you might be allowed to do the same kind of things with strings, lists, arrays, ordered pairs, and so forth. A kind of data item that can be moved around, stored, etc. in all of the usual ways is called a *first class* data item.

Although a language almost surely treats numbers, characters and a few other things as first class items, programming languages differ from one another in terms of which other kinds of things are first class. For example, C allows you to return a record (called a structure in C) from a function, but Pascal does not. C allows you to return some arrays from functions, but not others.

You are probably used to thinking of a function or procedure not as a data item, but as a tool that can be used in a program. But a real tool, such as a hammer, is a perfectly good thing, and there are things that you can do with a hammer, such as putting the hammer into a tool box, that do not involve using it as a tool. Some languages, including Python, Scheme and Cinnameg, allow you to treat functions and procedures as first-class data items. For example, a function can produce another function on the fly.

There is a surprisingly wide variety of things that have been made first class items in programming languages. Examples include variables, functions, procedures, expressions, types (as data items), modules, and even events that occur during the execution of the program, allowing a program to examine what it is doing or even what it might do in the future.

4.9. Persistent and ephemeral data

We have seen that the collection of operations that a given type of data supports has a strong influence on how you choose to represent that data. But the available operations can also influence your conceptual view. To illustrate, think of strings (lists of characters).

Suppose your operations only allow you to look at a string, and never to change it; for example, you can get the k-th character of a string, and you can concatenate two strings to build a new string, but no operation is available to change the k-th character of a string. Then you think of a string as a sequence of characters, and you can completely characterize a particular string by saying that it is "frog"; there is no extra hidden information about it. If variables x and y both contain "frog", then they *must* be equal, since there is no hidden information that could be used to distinguish them from one another.

But now suppose another operation is added that changes a string, modifying the memory that it occupies. For example, you might be allowed operation addChar(s, c) that adds character c to the end of string s. Then it is no longer adequate to describe a string by a particular sequence of characters; you have to say something more about it. For example, you can describe a string as an object that currently holds information "frog", but that might hold a different string in the future. The additional information is the memory address where the object is stored. Two objects that both currently contain "frog" might not be the same object. The conceptual view of a string has to adapt to the available operations.

Finally, suppose that you not only have operations that modify the characters in strings, but you also have the ability to use memory sharing. For example, suppose that

strings are represented sequentially, with an end marker, and that function tail(s) just gives you the memory address one character after the beginning of s, which is a representation of the tail of string s. Then sequence of operations

```
x = "frog";
y = tail(x);
addchar(x, 'g');
```

leaves variable y referring to string "rogg". A change to x has also changed y, because they share memory. The conceptual view of strings that you use must be accurate, and it must predict the behavior of programs like this. You are forced to adopt a view of strings that is very close to the way they are represented in the memory, including the memory sharing; the simple view that a string is a sequence of characters is untenable.

Generally, types of data that only support operations that look at the data are called *persistent*[2] or *immutable* types of data, and they tend to enjoy relatively simple conceptual views. Types of data that allow changes are called *ephemeral* or *mutable* types of data. Declarative languages typically only allow persistent data types, while imperative languages typically employ ephemeral data types. But there are exceptions. For example, Java, an imperative language, provides strings in a persistent or immutable way; no operations are available that change a string.

4.10. Exercises

4.1. What are the head and tail of each of the following lists? If the list does not have a head or tail, say so.

(a) $[2, 4, 6, 8, 10]$

(b) $[2, 7]$

(c) $[5]$

(d) $[\,]$

4.2. What is the difference between a simple value and complex value?

4.3. Integers written in decimal notation appear to be sequences of digits. Why are integers not usually considered to be complex values?

4.4. What is the difference between a symbol and a string?

4.5. Some languages allow variables to be first-class items. What does it mean to pass a variable to a subprogram? Give an example where it might make sense to use a variable as a data item in a program.

4.6. Give two important advantages of sequential representation of lists over linked representation.

[2]This definition of the word *persistent* is common to some fields, including analysis of algorithms. It is different from the way the same term is used in object-oriented programming, where a persistent object is one that can change over time, but that can be modeled at any instant as if it holds a particular value, so that it has meaning outside the context of a particular program. (An object that contains references to other objects is not generally persistent, since it is tied to the context where those other objects exist.) The same term is also used for data that is stored in persistent memory, such as on a disk, having nothing to do with the definition used here.

4.7. Give two important advantages of linked representation of lists over sequential representation.

4.8. When a data type is persistent, the presence of memory sharing does not affect your conceptual view of the data. Why not?

4.9. Suppose that a character occupies 2 bytes and a pointer occupies 8 bytes. Thinking of fairly long strings, how much more memory (as a factor) does the linked representation use to represent a string, compared to the sequential representation? What if the machine architecture requires memory addresses to be *aligned*, meaning that an 8 byte pointer must be stored at a memory address that is a multiple of 8?

4.10. Suppose that lists are represented sequentially in memory, and suppose for simplicity that lists are persistent, so that, once created, you will never change what is in a list. In order to pass a list to a subprogram, is it necessary to copy the entire list into the subprogram, or is there a more efficient way to pass the list? If there is a better way, what is it?

4.11. Java represents strings in sequential form. Operator + is string concatenation. If x is a string of length n and y is a string of length m, how long does it take to compute $x + y$, in terms of n and m, to within a constant factor?

4.12. Java does not provide any operations that modify strings. (You can compute new strings, but never change an existing one. For example, computing $s + t$, the concatenation of strings s and t, does not change s or t.) But Java has one unusual operator, ==, which asks whether two strings are stored in the same place in memory. A variable that holds a string actually holds the memory address where the string is stored. So if variables x and y both hold string "frog", then $x == y$ might be true and might be false, depending on whether the occurrences of "frog" referred to by x and y are in the same memory location. Explain how the presence of the == operator affects the conceptual view of a string. Is it adequate to explain a string solely as a sequence of characters?

4.13. For this question, assume that lists are persistent (a list cannot be modified once it is created), as is the case in Cinnameg. Memory sharing is allowed. Computing the length of a linked list normally requires counting the number of links that need to be traversed before the null pointer at the end of the list is seen. Describe a modification of linked lists where computing the length takes only a constant amount of time, regardless of how long the list is. Ensure that the operations of taking the head and tail, and of adding a value to the front of a list (producing a new, longer list), still take a constant amount of time.

4.14. Describe a scheme for representing lists such that the operations of indexing (taking the k-th member), getting the length, and computing the tail of a list each take a constant amount of time, independent of the size of the list. Notice that you do not need to build larger lists in constant time. Assume that memory sharing is allowed.

4.15. Describe a hybrid representation of lists that is linked, but allows an arbitrary number of items in each cell. Both standard linked representations and sequential representations are special cases of this. How would you find the k-th member of such a list?

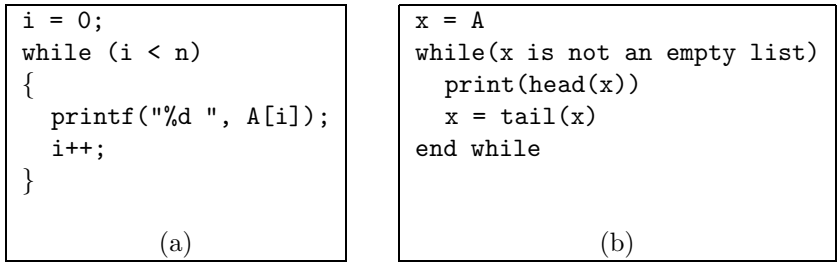


Figure 4.4: **Two program fragments to print members of a list A.** Program (a) is written in C and uses indexing as the fundamental method of getting a member of an array. (C begins indexing at 0, so an array A of size 4 has members $A[0]$, $A[1]$, $A[2]$ and $A[3]$.) That style is appropriate for a sequential representation of lists, where indexing is efficient. Program (b) is written in pseudo-code, in a style more appropriate for a linked representation, as is used by Lisp.

4.16. Suppose that lists are persistent. Memory sharing is allowed. Describe a version of the hybrid representation of the previous question that allows you to take the tail of a list in constant time, regardless of how many items are stored in the first cell of the list. How is the tail computed?

4.17. What motivation is there for using a hybrid representation for lists. What are the advantages? What are the pitfalls?

4.18. How long, as a function of n, does it take to run the loop of Figure 4.4(a)? Give the answer to within a constant factor (big-O notation).

(a) Assume arrays are stored using sequential representation.

(b) Assume arrays are stored using linked representation.

4.19. How long, as a function of n, does it take to run the loop of Figure 4.4(b)? Give the answer to within a constant factor.

(a) Assume that lists are stored using linked representation.

(b) Assume that lists are stored in sequential representation, with the size stored at the beginning. (How can the tail of such a list be computed? How long does it take?)

(c) Assume that lists are stored in sequential representation, with an end marker. (How can you take the tail? Assume that memory sharing is allowed.)

4.20. A sequence can be stored as a linked list with pointers going both directions, both from a cell to the next one, and from a cell to the previous one. Can you see any advantage to this representation? What kind of operation might be easier to implement?

4.21. Neither the linked representation nor the sequential representation is good for representing a list that can be extended cheaply by adding a member to its end. Describe

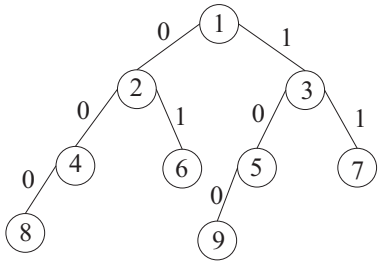


Figure 4.5: **A tree representation of a list.** This tree represents a list of nine values, with the k-th value stored at the node labeled k, for $k = 1, \ldots, 9$. (As shown, it represents the list $[1, 2, 3, 4, 5, 6, 7, 8, 9]$.)

a representation that does allow extending a list at its end cheaply (in a constant amount of time, regardless of the length of the list), while still allowing rapid access to the beginning of the list, and rapid removal of the first member. Assume that lists are ephemeral, and adding an item to the end of the list changes the list, destroying the information that was formerly held in the list.

4.22. Does your solution to Exercise 4.21 work when lists are required to be persistent, and adding a member to the end of a list creates a new list, leaving the old one intact? Did you need to destroy the old list in order to create the new one?

4.23. Modify your solution to Exercise 4.21 so that, in addition to extending the list, deleting the last member of the list is cheap, and getting the list of all members except the last is also cheap. Assume that lists can be modified.

4.24. You can represent a list as a pair (N, T) where N is the length of the list and T is a binary tree. In the tree, label each edge by either 0 or 1, choosing label 0 for each left edge and 1 for each right edge. See Figure 4.5. Each node in the tree has an associated binary number. To get it, start by writing 1. Now, starting at the given node, read the labels upwards to the root, writing each 0 and 1 encountered after the 1 that you started with. For example, if you start at the node labeled 6 in the tree shown in Figure 4.5, then you write 110, starting with 1 and then writing 10. Notice that 110 is the binary representation of 6. You store the sixth member of the list at the node labeled 6 in the figure. In general, you store the first member at the node labeled 1, the second member at the node labeled 2, etc.

Suppose that lists are persistent, so you are not allowed to destroy one list in order to build another one. Show how to use this tree representation to implement the following three operations in time proportional to $\log_2(n)$, where n is the length of the list. The operations are (1) indexing (finding the k-th member), (2) adding a member to the end of the list (producing a new list), and (3) removing the last member of the list (again, producing a new list). Explain how each operation is implemented by writing it in pseudo-code. Be sure to check that, after you do any given operation, the previous lists still exists.

4.11. Bibliographic notes

IEEE floating point numbers are described in [55]. Okasaki [76] shows how to implement lists efficiently as trees. Books on data structures, such as Carrano and Prichard [22], cover representations of sequences, trees and other data types, and efficient algorithms that work on those representations.

Wolfram [104] explains Mathematica. Schwartz et. al. [86] describe SETL. You can find information about Griffin at http://cs.nyu.edu/cs/faculty/goldberg/griffin.html. The language Tahiti [1] treats events as first-class data items.

Exercise 4.24 describes an implementation of *persistent arrays*. Aasa et. al. [2] compare some representation schemes for persistent arrays, and Paulson [78] describes implementing them in Standard ML.

Chapter 5

Imperative Programming

From the earliest days of programming directly in machine language to today's more advanced languages, imperative programming has held a dominant spot in software design. Imperative languages tend to be close, in the view that they present to a programmer, to the way a computer really works, and imperative programs can be highly efficient since there is little between the programmer and the computer to disrupt efficiency. A good C programmer can often visualize the machine language program into which his or her C program will be translated.

Imperative programming languages also enjoy the advantage that they are relatively easy to implement, since the basic operations are already provided by the computer hardware. As a consequence, language implementors have time available to work on fine-tuning their implementations to produce very efficient machine language programs, further improving efficiency.

5.1. Statements, variables and assignment

The basic unit of computation in an imperative program is the *statement*, which is a command that performs an action, such as changing the value of a variable within the program or changing something external to the program, such as the contents of a file on a disk. A statement that changes the value of a variable is called an *assignment statement*. Figure 5.1 shows assignment statements in a few different languages. For uniformity, we will write *var* := *expr* to indicate an assignment.

5.2. Control: early approaches

Each statement potentially affects the actions that are done by future statements. Assignment $x := 4$, for example, affects the value later computed by assignment $y := x - 2$. A program achieves its ends not only by choosing which statements to do but also by controlling the order in which they are done, or how control flows through the program.

An early method of specifying the flow of control was to use *flowgraphs*. Restricting attention to assignments, a flowgraph is a diagram made from a start node, a stop node and computational nodes of two kinds, linked by directed edges (arrows between the nodes). The first kind of computational node is an assignment node, usually drawn as a rectangle,

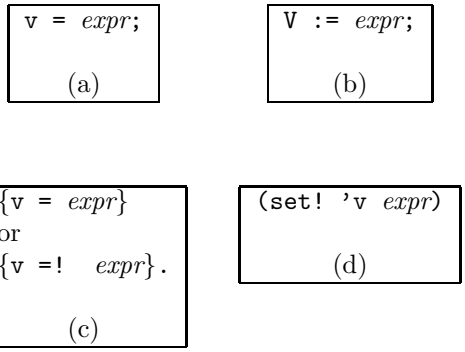


Figure 5.1: **Statements that assign the value of expression *expr* to variable v written in (a) C, (b) Ada, (c) Cinnameg and (d) Scheme.** In Cinnameg, you use `=` to create a variable and give it a value, and `=!` to change its value after it has already been created.

and holding an assignment statement. When the assignment node is reached during computation, the assignment statement inside it is performed. The node has one edge leading out of it, pointing to the node to continue with.

The second kind of computational node is a choice node, usually drawn as a diamond, and containing an expression that must produce a boolean value. It has two directed edges leaving it, one labeled true, the other labeled false. The next node to continue with is found by following the edge with the label that matches the value of the expression.

We have ignored issues concerning inputs to a flowgraph and outputs that it produces. For simplicity, presume that certain variables represent parameters, having values when the flowgraph starts, and that other variables represent results. Figure 5.2 shows an example of a flowgraph that computes $m = n!$. Variable n is the parameter and m is the result.

There is little doubt that flowgraphs are simple. Excluding the start and stop nodes, there are only two kinds of nodes, with simple rules for connecting them. Loops in the program are immediately evident as cycles in the flowgraph. Among other things, Occam's razor tells us to choose simple solutions when there is no compelling reason to select more complex ones. So why not program with flowgraphs?

Early programmers did just that, at least in their initial designs. Unfortunately, flowgraphs do not lend themselves well to typewriters, and early computers had little graphics capabilities. (Even on modern computers, programming directly with flowgraphs is cumbersome.) So programmers encoded a flowgraph in a more convenient form, where each node was given a name or number, and the statements in the flowgraph were written as a program, with their numbers attached as *labels*. Links between the nodes were described by "goto" statements. The number of gotos was reduced by adopting a rule that, in the absence of an explicit goto, control flows to the next statement in the program. This approach was adopted in early versions of the first high-level programming language, Fortran. Figure 5.3 shows a Fortran IV program fragment that implements the factorial flowgraph. It assumes that N already has a value, and it does not do anything with the result M except compute it.

That style of programming flourished for several years, but it was found to have

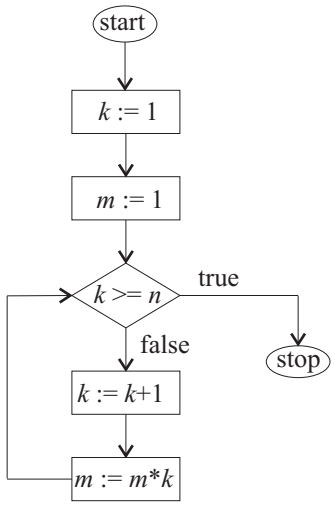


Figure 5.2: **A flowgraph computing m = n!.**

```
      K = 1
      M = 1
10    IF(K .GE. N) GOTO 20
      K = K + 1
      M = M * K
      GOTO 10
20    CONTINUE
```

Figure 5.3: **Computation of M = N! in Fortran IV.** This is a translation of the flowgraph shown in Figure 5.2. Statement labels are in the left margin; it is only necessary to label a statement that is the target of a goto. The CONTINUE statement is a "do nothing" statement that is just used as a place to put a label. The .GE. operator is Fortran's way of saying $\geq$.

two serious drawbacks. First, the program is much more difficult to read and follow than the flowgraph from which it is derived. Seeing a goto, the reader finds himself or herself searching up and down for the label. A second drawback is that there is almost no direct correspondence between the structure of the program and the structure of the computation that it describes. For example, a simple choice looks a lot like a loop to a quick glance. Two loops coming one after another can look a lot like two nested loops or even a program that just does choices and contains no loops, even though the structures of the computations are very different. Some programs made such complex use of gotos that they were nearly impossible to understand.

5.3. Structured programming constructs

To overcome the difficulties with direct translation of flowgraphs using gotos, a new form of programming language, a *structured programming language*, was developed. Flowgraphs and gotos were replaced by a small collection of *control structures* that indicate, in clear terms, how computation in a program flows. There are three fundamental kinds of control structures of structured programming: sequencing, choice and looping.

Sequencing

A sequence of statements is executed one after another, in the order written. For example, sequence

```
x := 0;
y := x + 1;
z := 2;
```

first sets x = 0, then sets y = 1 and last sets z = 2. This was, of course, already in use in Fortran IV. Structured programming simply recognized the importance of sequencing as a fundamental way of ordering computational steps.

Some languages, such as Ada and Cinnameg, allow sequencing in almost any context; anywhere you can write one statement, you can write several in a row. But some languages only allow sequencing inside a statement called a *compound statement*. In C, for example, a compound statement has the form $\{s_1 \ldots s_n\}$ where $s_1, \ldots, s_n$ are statements. The braces indicate the beginning and end of the sequence of statements.

Choice

A *conditional* statement allows a program to choose among alternatives. The most basic conditional statement in C has the form shown in Figure 5.4(a). Syntactic variations on conditional statements are common, but the general idea is similar regardless of the syntax. For example, Cinnameg uses the form shown in Figure 5.4(b).

Loops

A loop, also called an *iterative* statement, calls for performing a statement or a collection of statements repeatedly. The most common form of loop is a while loop, illustrated by Figure 5.6. There are syntactic variations on the while-loop, but the concept is the same in different languages.

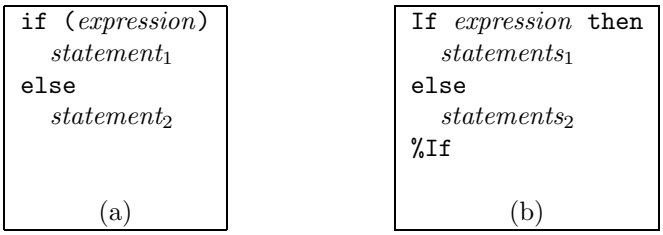


Figure 5.4: **Two syntactic forms of a conditional statement, (a) in C and (b) in Cinnameg.** Flowgraph 5.5 shows a corresponding flowgraph.

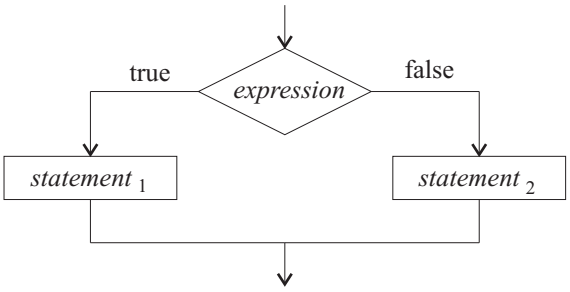


Figure 5.5: **A flowgraph of C statement** `if (`*expression*`)` *statement*$_1$ `else` *statement*$_2$ **.**

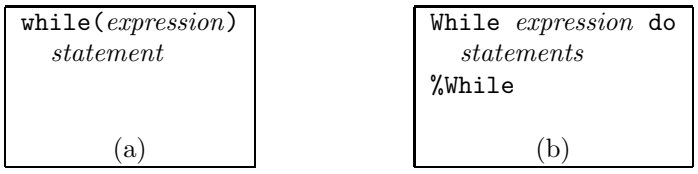


Figure 5.6: **While loops written in (a) C and (b) Cinnameg.** The meaning is the same as the flowgraph in Figure 5.7.

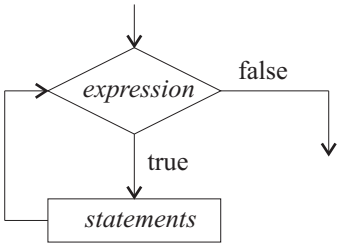


Figure 5.7: **A flowgraph of Cinnameg statement** `While` *expression* `do` *statements* `%While`**.**

```
if(x < 0)
  sign = -1;
else if (x == 0)
  sign = 0;
else
  sign = 1;
```

(a)

```
if X < 0 then
  Sign := -1;
elsif X = 0 then
  Sign := 0;
else
  Sign := 1;
end if
```

(b)

```
open Choose
  case x < 0  => {sign = -1}
  case x == 0 => {sign = 0}
  case x > 0  => {sign = 1}
%Choose
```

(c)

Figure 5.8: **A three-way choice written in (a) C, (b) Ada and (c) Cinnameg.** The Cinnameg form needs to be marked *open* to allow the bindings of sign to be visible outside the choose construct, which normally would be a block, as described in Section 6.1.

Contextual independence

Assigment statements, choices and loops all share the characteristic that they have a single entry point and a single exit point. So all can participate in sequencing. An important principle of structured programming is that any kind of statement can occur where any other kind can occur. There are no statements that have special privileges. The body of a while-loop can be (or can include) any kind of statement, whether it is an assignment statement, a compound statement, a conditional statement or another while loop. Also, the meaning of a construct does not depend on where it occurs. A while loop does not change its nature just because it occurs in a particular place in a program. Figure 6.1 shows a loop inside a loop.

5.4. Variations on choice

Multiway choice

It is common for a choice to have more than two alternatives. For example, consider the problem of computing the sign of a number, where the sign of x is defined by

$$\text{sign}(x) = \begin{cases} -1 & \text{when } x < 0 \\ 0 & \text{when } x = 0 \\ 1 & \text{when } x > 0 \end{cases}$$

There are naturally three alternatives in the definition of the sign function. The choices can be performed using if-then-else, as illustrated in Figure 5.8(a). A language with end markers can make multiway choices awkward because each two-way choice needs an end marker. Typically, a special provision to avoid excessive end markers is made for multiway choices, as is shown for Ada in Figure 5.8(b). Cinnameg provides a construct that is illustrated in Figure 5.8(c) for doing multiway choices.

```
switch(n)
{
case 0:
  printf("You are lucky\n");
  break;
case 1:
  printf("You will be happy\n");
  break;
case 2:
  printf("You will be rich\n");
}
```

(a)

```
Choose matching n
  case 0 =>
    Displayln "You are lucky".
  case 1 =>
    Displayln "You will be happy".
  case 2 =>
    Displayln "You will be rich".
%Choose
```

(b)

Figure 5.9: **A statement in (a) C and (b) Cinnameg that prints a fortune based on the value of variable n.**

Multiway choice based on a value

Sometimes, instead of testing several Boolean expressions, you want to base a decision on the value of some variable or expression. For example, you could do one thing if x is 1, another if x is 2, and yet anoter if x is 3. Figure 5.9 shows an example written in C (a switch statement) and Cinnameg (a variation on the choose construct).

There is an important difference, beyond mere syntax, between the Cinnameg choose construct and the C switch statement that tells you something about the design philosophies of those two languages. The C switch statement requires that the value being tested (n in Figure 5.9(a)) is an integer, or something that is represented by an integer, such as a character. It also requires the case labels to be constants, not variables or more general expressions. You are not allowed, for example, to use a case of the form `case k:` where k is a variable. Those restrictions allow the compiler to exploit features of computer architecture to perform the switch efficiently. The compiler can create an array of instruction addresses and select an address to jump to from that array based on the value being tested. For example, the array entry at index 1 would be the address of the machine-language code for the instructions to be performed when n is 1. That is consistent with the C philosophy of providing efficient and direct access to machine resources.

The Cinnameg **Choose** construct has no such restrictions. The philosophy of Cinnameg is to provide a more logical language, and allowing any kind of value to be tested is consistent with that view. That does not preclude using an efficient implementation when the compiler recognizes that the value being tested happens to be an integer, and the labels happen to be constants.

5.5. Variations on loops

Programs written using structured programming constructs tend to be much easier to read than programs written using gotos because they clearly state what they are trying to do. The (static) structure of the program reflects the (dynamic) structure of the computation.

```
Loop
  do Read a value
  exitcase the value is the end marker => ()
  loopcase else =>
    Process the value
%Loop
```

Figure 5.10: **A read/process loop in Cinnameg.** The loop has three kinds of cases: those that request computation to performed (**do**); those that exit the loop (**exitcase**); and those that continue the loop for another iteration (**loopcase**). There can be any number of cases, in any order, so the programmer can decide when to exit the loop. Statement () means *do nothing*.

But the common structured programming constructs do not do a good job of indicating what is being done in all cases. The while loop, for example, has an important defect; the decision about whether to continue the loop is made only at one place, the top of the loop. In practice, loops often do not fit that structure. For example, imagine a program that reads and processes a sequence of values until it reads a value that marks the end of the input. The loop is most naturally expressed as follows.

Loop
1. Read a value.
2. If the value is the end marker then exit the loop.
3. Process the value.

End loop

Notice that the decision about whether to continue the loop is made in the middle of the loop body, not at the top. When that is written as a while loop, the decision must be forced to the top of the loop. That can be done by duplicating the line that reads a value, and forcing each loop iteration to prepare for the next one, as follows.

Read a value.
While(the value is not the end marker)
  Process the value.
  Read a value.
End loop

The goal of structured programming is to make the form of a program clearly reflect its intent. By forcing the loop test to be made at the top, the while-loop has caused the program to migrate away from its intent.

Solutions to this problem have been provided in many languages. In C, for example, there is a **break** statement that can be used to terminate a while-loop at any point. But the while loop is intended to convey information to the reader that the exit test is at the top; a break statement makes a lie out of that. A better solution is a loop that is normally intended to exit at any point, and some languages provide those loops. Figure 5.10 shows the read/process loop in Cinnameg.

```
m = 1;
for(k = 2; k <= n; k = k + 1)
{
  m = m * k;
}
```

(a)

```
{m = 1}
For k from [2, ..., n] do
  {m =!  m * k}
%For
```

(b)

Figure 5.11: **Computation of $m = n!$ expressed in (a) C and (b) Cinnameg using a for-loop.** The C form contains an initialization part, a test to determine whether to keep going, and an update part, all together in the heading rather than spread around the loop. The Cinnameg for-loop performs a sequence of statements for each member of a list.

Managing the loop control variables

One goal of programming languages is to reduce the frequency of errors in programs. Among the constructs used in structured programming, loops tend to be among the most error-prone. In recognition of that, most programming languages that support loops offer a kind of loop that can reduce the error rate.

Typically, a loop has a small collection of variables, called the loop *control variables*, that are initialized at the start of the loop and are updated each time through. Common mistakes include failure to initialize and failure to update one or more of the control variables. A *for-loop* offers a solution when there is one control variable, or one primary control variable. The for-loop syntax either has special places for the initialization, test and update code, so that none of them can be accidentally left out (illustrated by C), or performs all initialization and updates automatically (illustrated by Cinnameg). Figure 5.11 shows an algorithm to compute $n!$ written in C and Cinnameg using a for-loop. It is almost impossible to forget to initialize or update the control variable k.

Unfortunately, it is still possible to forget to initialize or update m. Loops often have more than one control variable, and it would be nice if all of them could be managed in the loop heading.[1] Cinnameg's general loop allows you to specify several control variables, in a way that makes it difficult to forget to initialize or update any of them. The factorial loop in Cinnameg is shown in Figure 5.12.

5.6. Procedural programming and a fourth basic algorithmic construct

A procedural programmer is encouraged to invent his or her own kinds of statements and expressions. That is not to say that the programmer introduces new syntax into the language; very few languages allow that. Rather, the programmer creates *procedures* that have the effect of being new kinds of statements, and *functions* that have the effect of being new kinds of expressions. Procedures and functions are examples of *subprograms*.

[1] It is possible to have more than one control variable for a C for-loop. See exercise 5.3.

```
Loop* (k,m) = (1,1)
  exitcase k >= n  => m
  loopcase else    => Continue(k+1, m*(k+1)).
%Loop*
```

Figure 5.12: **The factorial computation expressed in Cinnameg.** The loop control variables are k and m and both are initialized to 1. A loop can be an expression if it starts with Loop*; each exitcase tells the value that the expression produces. When $k \geq n$, the loop exits and produces value m. Otherwise, the loop continues back to its top, replacing the values of k and m by $k + 1$ and $m * (k + 1)$, respectively. The loop syntax makes it impossible to omit the initialization or the update.

Suppose that a programmer wants the ability to sort an array of numbers into ascending order. Obviously, it would be nice to have a **sort** statement in the language, as COBOL does, but most languages do not have that. Instead, the programmer creates a sort procedure, and uses standard *procedure-call* syntax to use it. For example, to sort array A, you might write

`sort(`A`);`

5.6.1. Information flow

Flowgraphs are a way of describing *control flow* in programs, namely, how a thread of control weaves its way through the statements, performing them in a specific order. In a flowgraph, control can flow in arbitrary ways, since any node in the graph can conceivably point to any other node. Structured programming is much more restrictive (and, of course, structured) in terms of how it allows control to flow through a program. Each basic construct has a single entry point and a single exit point, and it is partly the restricted flow of control that makes structured programs easier to understand than flowgraphs with arrows pointing in arbitrary directions, obscuring what the flowgraph does.

Another kind of flow that occurs during the execution of a program is a flow of information. For example, a line of a program that stores a value into variable x is implicitly passing information to another line that uses that value of x. Understanding the flow of information in a program is just as important to a programmer as understanding the flow of control.

An obvious and simple way to pass information is to store it into a variable in one place and get it out of that variable in another place. On the scale of a single subprogram definition, that is generally how information flows. On a larger scale, that approach leads to the idea of shared, or *global*, variables, that can be used by more than one subprogram. Figure 5.13(a) shows a definition, written in C, of a procedure that prints the factorials of the numbers from 1 to 10, using a helper procedure that computes factorials. Information is passed through global variables factIn and factOut. But use of global variables allows unrestricted information flow, just as a flowgraph allows unrestricted control flow. When global variables are used, it is not clear, except after careful reading, how information flows in a program. In large programs it can be too complex to understand.

A more structured approach to information flow is to use *parameters* of subprograms, and sometimes to return results from functions. Figure 5.13(b) uses parameter passing to

```
int factIn, factOut;
void computeFactorial()
{
  int k = 1;
  factOut = 1;
  while(k < factIn)
  {
    k = k + 1;
    factOut = factOut * k;
  }
}

void PrintFactorials()
{
  int k = 1;
  while(k <= 10)
  {
    factIn = k;
    computeFactorial();
    printf("%i %i", k, factOut);
    k = k + 1;
  }
}
```

(a)

```
int factorial(int n)
{
  int k = 1;
  int m = 1;
  while(k < n)
  {
    k = k + 1;
    m = m * k;
  }
  return m;
}

void PrintFactorials()
{
  int k = 1;
  while(k <= 10)
  {
    printf("%i %i", k,
           factorial(k));
    k = k + 1;
  }
}
```

(b)

Figure 5.13: **Two implementations of a subprogram that prints the factorials of the numbers from 1 to 10, both written in C.** Version (a) uses global variables to pass information to and from the helper procedure, and version (b) passes information in a more structured way. The printf lines print two integers using formatted prints. Like the other subprograms here, printf is not built into the language, but is a procedure.

accomplish the same effect as Figure 5.13(a), but avoids global variables. With parameter passing, information flow is clear and controlled, and you can see just what is being passed to or from a subprogram by reading the program. So the (static) structure of the program tells you something about how information flows (dynamically) in the program while it runs, in a way that is similar to how the (static) structure of a loop or conditional statement tells you something about how control flows (dynamically) during program execution.

A big problem with use of global variables concerns whether a subprogram is *reentrant*. There are times when you need to use two copies of a given subprogram simultaneously, either because your program has more than one thread of control or because a subprogram either directly or indirectly calls itself. Using global variables for communication with a subprogram makes it difficult to have two copies of that subprogram running simultaneously without those copies interfering with one another, since the two will want to store different information in those shared variables. The same problem does not occur when parameters are passed because different copies of a subprogram store their information in different places.

In modern programming, parameter passing (and use of return values from functions) is considered the preferred way to send information either to or back from a subprogram. Chapter 8 discusses implementation of parameter passing using the run-time stack.

5.6.2. Syntactic approaches to parameter passing

Procedures typically require parameters to be written a specified order; there is a first parameter, a second, and so on. Suppose, for example, that Ada procedure *Simulate* has parameters called *Startval*, *Duration* and *Stepval*, all integers, in that order. Then to call Simulate, you would write

```
Simulate(0, 100, 2);
```

indicating that the Startval is 0, the Duration is 100 and the Stepval is 2. Unfortunately, programmers are rarely good at remembering the correct order, and it is easy to make a mistake, and write the parameters in the wrong order. Ada offers a way around that. You can include the parameter names in the call:

```
Simulate(Startval => 0, Duration => 100, Stepval => 2);
```

There are two advantages to that. One is that it is clear which parameter is which. A second is that the programmer does not need to remember the order of parameters. The compiler will reorder them if necessary. So

```
Simulate(Duration => 100, Stepval => 2, Startval => 0);
```

works perfectly well. The programming language Smalltalk takes the point of view that parameter names should always be given, and even drops the procedure name entirely, instead letting the parameter names play the role of procedure name. In Smalltalk, you might ask a particular object sim to perform procedure startval:duration:stepval: as follows.

```
sim startval:  0 duration:  100 stepval:  2.
```

Smalltalk does not allow you to reorder the parts. The procedure name is startval:duration:stepval:, not stepval:startval:duration:.

```
{sum(n) = ans |
  open Loop (s, k) = (0, 0)
    exitcase k == n =>
      {ans = s}
    loopcase else =>
      Continue (s + k, k + 1).
  %Loop
}
```

(a)

```
{sum(n) = ans |
  open If n == 0
    then {ans = 0}
    else {ans = sum(n-1) + n}
  %If
}
```

(b)

Figure 5.14: **Two implementations of a function sum(n) =** $1+2+\cdots n$**, written in Cinnameg.** Function (a) uses a loop and function (b) achieves similar control using recursion. If you use a run-time stack to allow multiple copies of a subprogram to be running simultaneously, as discussed in Chapter 8, then there is really no difference between recursion and subprogram calls in general. You know, for example, that sum(3) should produce 1 + 2 + 3 = 6. A call to sum(4) produces answer sum(3) + 4 = 6 + 4 = 10.

5.6.3. Recursion

Subprograms were introduced to programming languages as a way of providing procedural abstractions, allowing a programmer to add, in effect, new kinds of statements and expressions to the language. Subprograms also make it possible to break up a large program into smaller units that are easier to understand, and make it possible to provide prewritten libraries of subprograms.

But the introduction of subprograms also opens up the possibility of a new control mechanism, *recursion*. A subprogram can call itself, and the ability to do that can be used as a substitute for loops. Figure 5.14 shows two implementations of a function that computes the sum $1 + 2 + \cdots + n$, one using a loop and one using recursion. Recursion is explored in more detail in later chapters. But it is worth noting that, in order to make use of recursion, a subprogram must be reentrant, since two or more copies (the caller and the callee) will both be active at once. One will be waiting for the other to return, but it remains alive, even if it is paused.

5.7. Exercises

5.1. Explain why flowgraphs were replaced by structured programming constructs.

5.2. How does Cinnameg's **Loop** or **Loop*** construct differ from a while-loop, in ways other than obvious syntactic ones? What is the important difference to a programmer?

5.3. The C for-loop can be used with multiple control variables. In the loop heading `for(`A`;`B`;`C`)` you can use a comma to separate steps in individual parts. For example, `for(`A, B`; `C`; `D, E`)` says to do both A and B initially, in that order, and to perform both D and E at the end of each iteration.

(a) Show how to write the factorial function (Figure 5.11) in C with all of the computation done in the for-loop heading. Both k and m should be initialized and updated in the heading. The body of the loop should be an empty set of braces, indicating that nothing should be done in the body.

(b) Does this feature of C make it more difficult to forget to initialize or update one of the control variables? Explain.

(c) Do you believe this style of programming makes the loop easier to understand or more difficult to understand?

(d) The comma can actually be used in any expression; A, B says to compute A then B, with the value of B used as the expression's value. Novice C programmers sometimes try to return two values from a function by writing

```
return x,y;
```

What does this do? In your opinion, is this a difficulty with the language or just a difficulty with novice programmers?

5.4. Each case in a C switch statement is normally ended by a break statement. If the break statement is omitted, then control flows from one case directly into the body of the case that follows it. So, without the break, more than one case body is performed.

(a) Show how to write a C program that prints the first five verses of "The Twelve Days of Christmas" using a switch without any breaks. (Feel free to substitute a different but similar song that involves repetition of lines from prior verses.)

(b) Do you find the requirement for a break to be a desirable feature? Why or why not? Can you suggest a way to modify C so that the breaks do not normally need to be written, but so that control flow from one case into the next is still possible?

5.5. Write a program fragment involving a loop that computes $m = n^k$, where n and k are integer variables that have been given values and $k > 0$. Do not change n or k in the process. Write the fragment in C and in Cinnameg.

5.6. Draw a flowgraph of your loop from question 5.5.

5.7. Give an example of a looping computation that naturally exits at the bottom of the loop. If you know C, C++, C# or Java, give an example of a loop control structure that has an exit test at the bottom instead of at the top.

5.8. Some languages that support structured programming constructs, such as C, retain the ability to do a direct goto. In C, you label a statement by writing the label name and a colon in front of the statement. Then statement `goto lbl;` jumps to label lbl. For example,

```
here:
printf("I am happy\n");
goto here;
```

prints I am happy over and over until you stop the program.

(a) Translate the factorial fragment of Figure 5.3 into C, using gotos, and avoiding structured programming constructs.

(b) Your solution to part (a) is probably considerally more difficult to read than an implementation that uses a while loop. But can you think of any ways in which a goto might improve the readability of a program, and actually be preferable to using only structured programming concepts? If you cannot, you might consult Knuth [59].

(c) It can be difficult to say just what a goto means when it is combined with structured programming constructs. Give an example where a goto jumps from outside a construct into the interior of it, where the meaning is unclear. Just where does the confusion arise? Do you think a goto should be allowed to jump into a construct from outside of it?

5.9. What are advantages of Ada's syntax for labeling actual parameters in a procedure call?

5.10. Explain why passing information to or from a subprogram by putting it into global variables is usually considered poor practice. Why do you think languages like C++ allow global variables?

5.11. (This requires some experience with object-oriented programming) Can you pass information implicitly between methods that belong to an object without using global variables or parameters? Explain how. Do you consider this a good way to pass information? Why or why not?

5.8. Bibliographic notes

Dahl, Dijkstra and Hoare [33] advocate the use of structured programming, and Dijkstra [36] recommends abandoning the idea of a goto statement. Knuth [59] offers an opinion that gotos can be combined reasonably with structured programming constructs. Wulf [105] argues that global variables can be harmful to program design.

Chapter 6

Variables and Scope

6.1. Scope

A name can be *bound* to variables, subprograms, etc. That part of a program where a binding of a name is visible is called the *scope* of the binding. We look only at scope on the scale of a single subprogram definition of similar unit of a program here.

Details of the rules for determining scope differ from one language to another, but there are some general similarities. Most languages have some notion of a scope unit, consisting of a section of a program that we will call a *block*. The *lexical scope rule* indicates that, when an identifier is bound, its binding is visible only within the innermost block that contains the binding. Typically, the binding is further restricted to be visible only from the point where the binding is made to the end of the innermost block that contains the binding; but some languages allow a binding to be visible, in some circumstances, before the point where the binding is made. In either case, visibility under lexical scoping is determined by the (static) structure of the program.

In C, for example, a block is either an entire subprogram definition or any part of a subprogram that is enclosed in braces ($\{\ldots\}$). Figure 6.1 shows a definition of subprogram *sort* written in C. In sort, five local identifiers, a, n, i, j and t, are bound; a is bound to (or names) an array and each of the others to a variable. None of the local bindings are visible outside the definition of sort, since the entire subprogram is a block. The scope of the bindings of a and n is the entire subprogram. The binding of i is visible in the entire inside of the definition. The scope of the binding of j is the body of the outer while-loop, and the scope of the binding of t is the body of the inner while-loop.

It is common convention to refer to the scope of a binding of an identifier as the scope of the identifier itself. So you might say that the scope of t in procedure sort is the inside of the inner while-loop. Of course, it is the *binding* whose scope you are really talking about. Such twists of terminology are common when they make the terminology less cumbersome, but you need to realize what is really being said.

Nested subprograms

Some languages allow a subprogram definition to be written inside another subprogram definition. Figure 6.2 shows an example in Pascal. Notice that the nested subprograms can make use of bindings that are in scope at the point where they are defined. For example, the

```
void sort(int a[], int n)
{
  int i = 1;
  while(i < n) {
    int j = i;
    while(j > 0 && a[j] < a[j-1]) {
      int t = a[i];
      a[i] = a[j];
      a[j] = t;
      j = j - 1;
    }
    i = i + 1;
  }
}
```

Figure 6.1: **A C definition of a procedure that sorts an array, given the array and its size.** There are four blocks, one being the entire definition, and the three others the sections delimited by $\{\ldots\}$. The blocks overlap one another, since one block is inside another.

definition of swapA occurs in the scope of A. So the scope of A extends across the definition of swapA, and swapA can use A.

Shadowing

Suppose that you attempt to rebind an identifier within its scope. That is not the same as changing the value of a variable. Think, instead, of binding an identifier to a different variable. What should happen? Typically, the old binding is not replaced, but instead is hidden, or *shadowed*, by the new binding, leaving a hole in the scope of the old binding. The C program fragment in Figure 6.3 illustrates that. The inner block contains a binding of x that shadows the binding of x in the outer block, and, within the inner block, identifier x is presumed to refer to the variable called x in that block. So the scope of the outer x has a hole in it, where the shadowing occurs.

Shadowing is usually not a good idea. If you have two variables with the same name, you can easily become confused. When a person reads a program, he or she cannot be expected to see every aspect of the program, and might not notice that a binding becomes shadowed. So shadowing can be misleading. Some languages, including Java and Cinnameg, disallow shadowing of one local binding by another.

Neither Java nor Cinnameg forbids shadowing a global binding (one done outside a subprogram) by a local one. Forbidding all shadowing would create a problem. Most programs make use of libraries, and a programmer rarely knows all things that belong to a library. If shadowing of the bindings in the library is not permitted, then a programmer must be sure to avoid using names of things that he or she is not even aware exist. Even worse, a library might grow over time as new functions and procedures are added, so the programmer would need to avoid names that might be added to the library in the future! For precisely this reason, most languages allow some shadowing. Extensive use of shadowing, however, is usually strongly discouraged when it can be avoided, so that human readers will

```
type intarray = array[1..100] of integer;
procedure sort(A: intarray);
  var k: integer;
  procedure swapA(var i,j: integer);
  var temp: integer;
  begin
    temp := A[i];
    A[i] := A[j];
    A[j] := temp
  end;
  procedure insert(n: integer);
  var i: integer;
  begin
    i := n;
    while (i > 1) and (A[i] < A[i-1]) do
    begin
      swapA(i, i-1);
      i := i - 1
    end
  end;
  begin
    for k := 2 to 100 do insert(k)
  end;
```

Figure 6.2: **An example of nested subprograms, written in Pascal.** Procedure sort sorts an array of 100 integers. It contains definitions of procedures swapA and insert, whose scope is restricted to the interior of the sort procedure. Notice that the nested procedures can make use of A, since it is in scope at the point where they are defined. (The word **var** in the heading of swap indicates call-by-reference, discussed in Chapter 5.)

```
{
  int x = 3;
  int y;
  {
    int x;
    ...
    x = 2;
    ...
  }
  print(x);
}
```

Figure 6.3: **A program fragment written in C that illustrates shadowing.** The binding of x in the inner block shadows the binding of x in the outer block. The scope of the binding of x in the outer block has a hole in it. The last line prints 3, since it is not in the hole, and so prints the value of the outer x.

not be confused.

6.2. Variables as data items

Variables, as we have used them up to now, are not first class items. Our programs talk about the value of a given variable, but not the variable itself. Our variables have just been names, and assigning a value to a variable is nothing but binding the name to a new value. Now, we turn to the possibility of letting a variable be a data item, where you can store a variable inside another variable, pass a variable as a parameter to a subprogram, or return a variable from a subprogram.

We will call a first class variable a *box*. You can think of it as like a shoe box that is capable of holding one thing. You can open the box and see what is inside, or you can take what is in it out, and put a new thing in it. But you can also just treat the box itself as a thing. You can, for example, give the box to someone else.

The box analogy suffers from a defect: in the real world, there is just one of each box. But in a program, you can put the same box in several different places. That defect can be eliminated by imagining a rack of numbered shoe boxes. When you say that you have a box, you actually only have its number, and when you lend the box to somebody else, you just give that other person the box number. Anybody can go to the rack and open a given box, either just looking at what is inside or replacing the contents of the box. You can imagine two boxes A and B that hold the same box C inside them. What they really hold is the number of box C. The rack of boxes is analogous to the computer's memory, and the box numbers are memory addresses, also called *pointers*.

Objects

Looking ahead to object-based programming, a box is a very simple kind of *object*. Generally, an object holds some information, and has some capabilities for manipulating that

```
{factorial(n) = @m |
  {k = [:1:]}
  {m = [:1:]}
  While @k < n do
    {@k =! @k + 1}
    {@m =! @m * @k}
  %While
}
```

Figure 6.4: **A factorial function written in Cinnameg, using boxes.** The end (`@m |`) of the heading line indicates that the value to be returned is the content of box m, after the function body is finished. Notice that the parameter, n, is not a box, but is a number.

information. A box holds just one item. Its capabilities include an operation to get the item, and one to replace the item with another one.

Cinnameg treats a box as an object. Expression `[:` x `:]` produces a new box, initially holding x. Operator @ gets the content of a box, and statement {`@`b `=!` v} stores v into box b. For example,

```
{b = [:0:]}
{x = @b}
{@b =! 5}
{y = @b}
```

makes b name a box, x name the number 0 and y name the number 5. Fetching the content of a box is called *dereferencing* the box, and @ is the Cinnameg dereference operator.

Notice that a box is anonymous; it has no name associated with it. But you often want to use a name to refer to a particular box, and it is common to create a box and give it a name in a single statement, as is done for b above. Figure 6.4 shows a definition of the factorial function, using boxes.

6.3. Expression context: lvalue and rvalue

C++ supports boxes, but with an important difference from Cinnameg. Consider C++ fragment

```
int num = 0;
num = num + 1;
```

When we viewed variables as names, we would say that this makes the name num refer to 0, and then changes the binding of num to 1. In reality, that is not what happens in a C++ program. The first line, `int num = 0`, creates a new box, initially holding 0, and names that box num. The second line, `num = num + 1`, changes the content of the box, not the binding of num (which still refers to the same box).

But if num is a box, then expression `num+1` appears to make no sense. You want to add 1 to the content of the box, not to the box itself. If C++ required the @ operator, then a more sensible assignment statement might be `num = @num + 1`. (Notice that it would

not make sense to write `@num = @num + 1` since, if box num currently contains 24, you do not want to perform statement `24 = 24 + 1`. On the left-hand side of the assignment statement, you really want the box.)

Programmers generally find the concept of rebinding names appealing, because there is only one level of binding (a name to a value) to think about, rather than two (a name to a box and a box to a value). When a C++ programmer writes $k = k + 1$, he or she usually thinks in terms of changing the value of k — that is, in terms of changing what the name k is bound to. That is not what really happens, but the language designers want to make it *appear* that the name k is being rebound. To achieve that fiction, the language definition must call for the dereference operation to be inserted automatically sometimes, but not all of the time. A C++ compiler must realize that the statement written $k = k+1$ really means $k = @k + 1$ where @ has been inserted before the second occurrence of k, but not before the first occurrence.

The language definition needs to include a notion of the *context* in which an expression occurs. There are two contexts: left context, which is like the left-hand side of an assignment statement; and right context, which is like the right-hand side of an assignment. Each expression has two values. The *lvalue* of expression E is the value that E has when it occurs in a left context. For example, if b is the name of a box, then the lvalue of b is the box itself. The *rvalue* of E is the value of expression E when it occurs in a right context; when E is the name of a box, the rvalue of E is the content of that box.

Context is not always obvious. For example, $A[k]$ is the k-th variable in array A, and in C++ assignment statement `A[k] = r`, k occurs in a right context, because you want its value, even though k occurs on the left-hand side of the assignment statement. For example, if box k currently holds 2, then $A[k] = r$ is equivalent, *at that particular use*, to $A[2] = r$.

6.3.1. Extending context to collections

The concepts of assignment, left context and right context can be extended to structured items that contains boxes within them. For example, if u and v are boxes then assignment $(u, v) :=$ (1,2) seems to make sense; it makes u hold 1 and v hold 2.

C++ and Java both have the notion of an object, which is a collection of variables along with procedures and functions (called methods) that work with those variables. For our purposes here, we only care about the variables, not the methods, so (for now) we model an object as a tuple of boxes.

C++ and Java treat assignments involving objects differently. Look at Java first. It treats objects as tight encapsulations. Suppose that object U is tuple (u_1, u_2) and object V is tuple (v_1, v_2), where u_1, u_2, v_1 and v_2 are all boxes. In Java, you refer to an object by putting it into a variable. So suppose that variable u contains object U, and variable v contains object V. Assignment $u = v$ just puts the object V held in variable v into u. It does not change any of u_1, u_2, v_1 or v_2, but instead just changes the object to which u refers.

But C++ treats objects as looser encapsulations, making their structures more visible. In C++, assignment statement $u = v$ has the same effect as doing two assignments, $u_1 = v_1$ and $u_2 = v_2$. That is, not only is v in a right context, so that it stands for its content (v_1, v_2), but the component variables v_1 and v_2 are also in a right content, and their contents are obtained. Similarly, putting u in a left context implicitly puts its component variables u_1 and u_2 into a left context. The C++ rule for assignments on objects is called *coherent assignment*.

An array is, conceptually, a list of boxes. Although Java and C++ treat assignments involving objects differently, they treat arrays in the same way. Suppose that u and v are variables that contain arrays. In both Java and C++, assignment $u = v$ makes variable u hold the same array as v. It does not change what is in either of the arrays. But Pascal treats assignments involving arrays more like the way C++ treats assignments involving objects, using coherent assignment. If u and v are (different) arrays, then Pascal statement $u := v$ copies the content of each box of v into the corresponding box in u. Afterwards, u and v are still different arrays, but they have the same values in them.

Cinnameg relies on the kind of statement that you perform to indicate how to treat things. If v is a pair of boxes, then statement $\{u$ `=` $v\}$ makes u be the same pair of boxes. To request a coherent assignment, you use a Copy statement. Generally, Copy $u = v$ %Copy copies from boxes in v to corresponding boxes in u, looking inside tuples and lists to find the correspondences.

It should be clear that, no matter what the standard or default definition of assignment is in a language, the programmer would ideally like to decide, on a case-by-case basis, just what assignment should mean. We have said that C++ performs assignment on objects in a coherent manner. But that is really just the default. C++ allows the programmer to define assignment in a different way for each type of object, by treating the assignment operator = as a function that the programmer can redefine, with a different definition for each type. (So operator = is *overloaded*.) For example, by defining $y = x$ to be the same as $y =$ copyList(x), when x and y are linked lists, you make assignments on linked lists copy the list. The following C++ definition does the job, assuming that a linked list has type List.

```
List& operator=(List& y, List& x)
{
  y = copyList(x);
  return y;      // Return the variable that was just changed.
}
```

6.4. Parameter passing modes

When a subprogram is defined, the definition needs a way to refer to the parameters that are passed to it. Usually, it gives names to the parameters. Those names, occurring in the subprogram definition, are called the *formal parameters* of the subprogram. If sqr is defined by sqr(x) = x*x, then x is the formal parameter.

When a subprogram is used, particular items, called the *actual parameters*, or sometimes the *arguments*, are passed to it as its parameters. If you compute sqr(6), then 6 is the actual parameter that is passed to sqr (and referred to by sqr as x).

6.4.1. Call-by-value

When a parameter is passed to a subprogram, the question arises whether you intend to pass the rvalue or the lvalue of the parameter expression. For example, when the parameter is the name of a box, do you want to send the content of the box or the box itself? In C, the answer is simple, since only one option is available: you pass the rvalue of the expression. That is called *call-by-value*. For example, C function sqr defined by

```
void valincrement(int* p)
{
  *p = *p + 1;
}
```

(a)

```
void refincrement(int& x)
{
  x = x + 1;
}
```

(b)

Figure 6.5: (a) C function valincrement increments a variable, using call-by-value. Type int* is the type of an address of an integer variable, or of a box that holds an integer, and * is the C dereference operator. A similar function (b) in C++ uses call-by-reference, indicated by type int&. The return type of both is void, indicating that no value is returned.

```
int sqr(int x)
{
  return x * x;
}
```

takes an integer x that is passed to it using call-by-value, and produces an integer result. Function sqr treats formal parameter x as a variable that receives an initial value at the beginning of the function call. If you write

```
z = sqr(y);
```

then the rvalue of y is placed into sqr's variable x and function sqr runs.

If you want to pass the lvalue of a variable to a subprogram in C, you must explicitly force the lvalue to be used. Operator & does that; it changes the context from right to left, requesting the lvalue of the expression. For example, procedure valincrement, shown in Figure 6.5(a), adds one to a variable. Formal parameter p is a pointer (the box). To pass it variable n (not the value of n), you can write

```
valincrement(&n);
```

Inside valincrement you need to remember that p itself is a variable, and when you get the value of p, what you really get is the box (such as n) that was passed to valincrement. To get the content of n, you need to add a second, explicit, dereference (* in C). When you want to change n, you need to change $*p$, not p. (Changing p would make p hold a different box.) So the extra * occurs in both left and right contexts.

6.4.2. Call-by-reference

An alternative convention is to pass the lvalue of the parameter expression implicitly. Fortran IV, for example, always does that. Passing the lvalue of the parameter is called *call-by-reference*.

C++ supports call-by-reference as an option, indicated by &, and Figure 6.5(b) shows an example. When refincrement is called, the lvalue of its actual parameter is passed, not the rvalue. Since the parameter now contains a box, extra dereferences need to be done. But those dereferences are implicitly inserted by the compiler when call-by-reference is used.

```
void alias(int& x, int& y)
{
  x = 1;
  y = 1;
  x = x + 1;
  y = y + 1;
}
```

Figure 6.6: **A C++ procedure that illustrates aliasing.** Statement alias(z,z) causes formal parameters x and y to be the same variable, and at the end of alias, both x and y will hold 3.

The extra dereference is a very real thing in a program; it signifies a machine instruction that performs a memory fetch, and makes access to reference parameters a little slower than access to local variables.

6.4.3. Aliasing

Suppose that statements

```
x = 1;
y = 1;
x = x + 1;
y = y + 1;
```

occur in a C++ program, and that variables x and y each hold integers. Can you say, regardless of the context in which these statements occur, what the values of x and y are after these statements? It turns out that the answer is no. The problem is that, in some contexts, x and y might have been declared to be references, and it is possible for them to refer to the same variable. Implicit dereferencing hides this phenomenon, known as *aliasing*. Figure 6.6 shows a procedure, alias, that illustrates.

Aliasing can be a serious problem, primarily because programmers are accustomed to adopting the convenient, but fictional, view that it is the names that are being bound, in which case aliasing can never occur. Programmers are likely to assume that there will be no aliasing, even in contexts where aliasing is a possibility.

6.4.4. Copying in and out

When a parameter is passed by value, the value is copied into a subprogram's formal parameter variable when the subprogram is called. For that reason, call-by-value is also called *call-by-copy-in*. For example, when sqr(5) is called, the value 5 is copied into the parameter x of sqr.

There is a complementary mode, *call-by-copy-out*, that is used to copy a value out of a subprogram and into a box owned by another subprogram. Like call-by-reference, it can only be used when the actual parameter is a box. When the subprogram is finished, the value of the formal parameter variable is copied out into that box.

Ada provides an example. Figure 6.7 shows an Ada procedure that computes the sum and difference of two integer values. Ada provides three parameter passing modes, **in**,

```
procedure Sumdiff(X,Y: in integer; Sum,Diff: out integer) is
begin
  Sum := X + Y;
  Diff := X - Y;
end sumdiff;
```

Figure 6.7: **An Ada definition of a procedure to compute the sum and difference of two numbers.** The parameter passing modes are expressed in logical terms **in** and **out**. The actual modes used are typically copy-in and copy-out.

out and **in out**, to tell how parameters are to work, where **in** parameters are only copied in, **out** parameters are only copied out, and **in out** parameters are copied in at the start and out at the end. When statement

```
Sumdiff(A,B,S,D);
```

is performed, the values of A and B are copied in to parameters X and Y, respectively. When Sumdiff is finished, the values of variables Sum and Diff are copied out to variables S and D, respectively.

6.4.5. Mechanism and intent

Notice that call-by-reference could be used instead of copying in or out. Both mechanisms get information into and out of a subprogram. Call-by-reference is more efficient than copying for large data structures, such as arrays, since all that is actually passed is the memory address of the data structure. But call-by-reference is less efficient than copying for small things, such as integers, because of the extra implicit dereferences that must be done each time the parameter is used.

It would be possible for Ada to provide call-by-reference as an alternative, since it is more efficient sometimes. But why should the programmer have to be concerned with which is more efficient? That really should be the compiler's responsibility. What the programmer wants to concentrate on is *how* the information flows; is it just into the subprogram, just out, or both in and out? The *mechanism* for accomplishing the programmer's intent is a separate issue.

An obvious thing to do would be to let the compiler decide to use copying for small things and call-by-reference for large things. Ada does just that. More precisely, the definition of Ada gives the compiler implementor the option to use copying or call-by-reference for *all* parameter passing modes, even mode **in**. Typically, implementors choose copying for small things, but they are not required to. Nor are they required to use references for large things, but they typically do. Ada distinguishes the semantics of parameter passing (the intent) from its implementation (the mechanism).

6.5. Exercises

6.1. Like C++, C uses boxes and implicit dereferencing, with left and right context. Rewrite the C program fragment of Figure 5.11(a) as it would be if there were no

notion of left and right context, and all dereference operations were required to be explicit. Use @ as the name of the dereference operator.

6.2. Rewrite the procedure definition of Figure 6.1 as it would be if there were no notion of left and right context, and all dereference operations were required to be explicit. Use @ as the name of the dereference operator. In C++, function parameters are boxes, and the function body can change their contents. Parameter a of function sort is actually a box that holds, as its content, an array. (If you want to, you can store a different array in that box, though few programmers would do that.) So, each time a is mentioned, you will need to do a dereference to get the array stored in that box.

6.3. The meaning of a fundamental language unit such as an expression or a statement should not depend on the context in which it occurs. From a syntactic standpoint, you can write $x + y$ in any context where an expression is expected, and, from a semantic standpoint, you should expect $x + y$ to have the same meaning in every context.

Having left and right context can be seen as a violation of this rule. Choose a point of view, and argue either that having left and right context really does violate that rule, or that having those two contexts does not cause any problems. Use examples from a familiar programming language to make your point.

6.4. C++ allows the programmer to redefine the meaning of assignment for some types, but not for all. For example, you cannot redefine how copying is done for integers or pointers. Do you consider that a reasonable restriction, or should it be done away with? Why do you think that restriction was made?

6.5. Using function valincrement from Figure 6.5, carefully show an evaluation of valincrement(&frost) where frost is a variable that holds 4 just before running this statement.

6.6. Using function refincrement from Figure 6.5, carefully show an evaluation of refincrement(frost) where frost is a variable that holds 4 just before running this statement.

6.7. Fortran IV uses only call-by-reference. How can a compiler deal with a parameter that does not have an lvalue? For example, how can a statement JUMP(2) be compiled, using call-by-reference?

6.8. A C++ subprogram is allowed to alter a parameter that is passed by value. The parameter is just a local variable that happens to be initialized to hold the actual parameter, and the subprogram can change what is in that local variable. An Ada subprogram, on the other hand, is not allowed to alter the value of a parameter whose calling mode is **in**. Explain why it is important for Ada subprograms not to allow modification of in-parameters. What could happen if an Ada procedure were allowed to modify a parameter that is only passed into the procedure?

6.9. The definition of Ada does not tell the programmer whether parameters will be passed by copying or by reference. It is possible for the behavior of a program to depend on which is done. Clearly, a programmer should avoid writing anything that depends on unspecified aspects of the semantics. Show how to write an Ada

subprogram whose behavior depends on the mechanism that is used for parameter passing. Do not worry about the details of Ada syntax, just use generic syntax. You will need to use parameters that are passed both in and out.

Chapter 7

Implementation of Programming Languages

7.1. Compilers

The only programming language that runs directly on a computer is that computer's machine language. All other languages require some kind of support: the language implementation.

One way to implement a language is to provide a compiler, which translates a program from one language (the *source* language) to another (the *target* language). Compilers have a central position in software design since the vast majority of the machine-language programs in existence today were produced by compilers, as translations of programs written in higher level languages.

Compilers have been written for hundreds of different source languages. Compilers typically translate to machine language or to assembly language, a symbolic form of machine language. But some compilers translate from one higher level language to another. For example, you can find compilers for Pascal or Common Lisp that translate into C. To use them, you must take the C program that is produced and translate that to machine language using a C compiler.

7.2. Linkers

Large programs are rarely written as one large piece, but instead consist of several separate *modules*, each of which is translated separately into machine language by a compiler. So the output of a compiler is not really a full program, but is a piece of a program. You might expect that, to put the pieces together into a complete program, you just put them one after another into a single file. Unfortunately, that does not do the job. The problem is that each module can refer to things that are defined in other modules. For example, suppose that module A needs to use subprogram EXPLODE, created in module B. Module A will have an instruction that needs to contain the memory address where the machine-language instructions for EXPLODE is stored. But when the compiler is translating module A, it does not know that address, since it is defined in another module.

To deal with that, the compiler creates a machine language instruction with the address missing. It also adds information to the translated program indicating where those

missing addresses are located, and which addresses they need installed. Machine language programs with this extra *linking* information in them are called *object-language* programs.[1] A tool called a *linker* puts several object-language programs together, replacing missing addresses by their correct values. For example, when the linker reads the translation of module A, it sees that there is a hole that must be filled in with the address of EXPLODE. When it reads module B, the linking information tells the linker where EXPLODE is located. The linker goes back to module A and inserts the correct address.

A linker can easily combine a program with others that were compiled earlier. Those *library* files provide assorted subprograms that are of general use to programmers, from relatively mundane functions such as one that computes square roots to much more involved ones that render graphics on the screen.

There are two general kinds of linking, static and dynamic. A *static linker* gets all of the subprograms, variables and other things from the libraries and combines them with the rest of the program into a single file. It has the advantage that the final product is a self-contained program, with all of the support that it needs built into itself. A *dynamic linker* avoids copying functions and procedures into a program, but instead copies them (from the library) either when the program is loaded into memory or by inserting *stubs* in place of each library subprogram. When it runs, the stub causes the library file to be loaded, and replaces itself by a reference to the library function.

Dynamic linking is typically preferred. It has the advantage that, when a dynamically linked library module is loaded, the operating system can store just one copy of it in memory, even if several concurrently running programs need to use it. Also, some library modules, not being needed immediately (or possibly at all if the features that they provide are not used), are not loaded right away, speeding up the startup of a program.

7.3. Interpreters

Another way to implement a programming language is via an *interpreter*, which works very differently from a compiler. It reads one source language instruction at a time and performs the action called for by that instruction immediately. For example, after reading `x = 4`, it finds where it is storing variable x and makes that variable hold 4. No translation to another language is done. An interpreter is in a loop, reading and performing instructions, as follows.

1. Get the current instruction.
2. Determine the meaning of the instruction.
3. Perform the action called for by the instruction.
4. Prepare for getting the next instruction.

Imagine that you have purchased a bicycle that requires assembly. A metaphor for a program is a set of assembly instructions written in Japanese (assuming that you do not read Japanese). In this metaphor, a compiler is a person who translates the instructions from Japanese to English. Once the translation is done, the translator leaves. It is then up to someone who understands English to assemble the bicycle. There are two stages: translation and execution. In the bicycle metaphor, an interpreter is someone who knows Japanese and assembles the bicycle for you, following the instructions. With this approach, there is only one stage: ask for the bicycle to be assembled.

[1] The term *object language* is an old one, and has nothing to do with the more recent notion of object-oriented programming.

7.4. Comparison of compilers and interpreters

Compilers have several advantages over interpreters. A big one is efficiency. After you have compiled your program, the compiled version will typically run from one or three orders of magnitude faster than will an interpreter running the same program. The interpreter is slowed down because it spends most of its time finding instructions, deciding what to do, finding the appropriate data to work on and checking that everything is going correctly. Only a small percentage of the time is spent doing the actual work. The interpreter might need to perform one hundred of its instructions to simulate one instruction of the source language.

Another advantage is *chaining* of translations. Some compilers translate into languages other than machine language. Older compilers for the language ML compile into into Common Lisp. You must then translate the Common Lisp program to another language. If you translate from Common Lisp to C, you would need to translate the C program to machine language. This chaining of translators may seem awkward, but it works reasonably well, and the program obtained in the end can be quite reasonable in terms of speed. Imagine the same thing with an interpreter. If each interpreter introduces two orders of magnitude slowdown, as compared to machine language, then running an ML interpreter via a Lisp interpreter would introduce four orders of magnitude slowdown.

Yet another advantage of a compiler is that the compiler does not need to be present when the compiled program is run. You can sell a machine language program, translated from C, to somebody who does not have a C compiler. You might even be able to run the machine language program on a computer that is too small to support a C compiler. (Imagine writing a program to control a toaster.) In contrast, dialects of Basic are typically implemented using an interpreter. If you sell a Basic program to somebody, your customer will be unable to run the program without a suitable Basic interpreter to perform the actions called for by the program.

The advantages offered by compilers are compelling. So why would anybody choose to create an interpreter? One reason is that it is typically easier and cheaper to create an interpreter. So if you want to implement a new language, and are on a tight budget, an interpreter might be a good way to go. Another issue is portability. There are two aspects to portability for an interpreter. First, an interpreted program will run on any machine that has the interpreter, but if you sell a machine language program, it will only work on machines that support that machine language. Java is typically (but not exclusively) implemented by an interpreter, making Java programs highly portable. Second, the interpreter itself is portable, and can often be moved from one machine to another by simply recompiling it for the new machine. A compiler must be retargeted to produce machine language for the new machine.

Security can also fall on the side of interpreters. An interpreter can contain instructions to check what it is doing, both for the purpose of ensuring that nothing very bad is being done, and to ensure that a program stays within a set of security guidelines. Implementations of Java, for example, allow you to set security policies, limiting what a Java program is allowed to do.

The prize for efficiency usually goes to compilers, but that is not always the case. The cost of running the compiler needs to be taken into account. If you intend to compile a program once and run it many times, then the cost of compiling is irrelevant. But if you intend to run a program only once, as you might while testing or for prototyping, it can take more time to compile a program than to complete running it using an interpreter. For

those purposes, interpreters can be preferable.

7.5. Hybrid implementations

Most interpreters are actually not pure interpreters, but are combinations of interpreters and compilers. A compiler is used to translate a program from the source language into some intermediate language that has been chosen to be easy to implement with an interpreter. Intermediate languages are often called *byte codes* because their instructions typically begin with a byte that tells the kind of instruction. A byte code is thought of as the machine language of an *abstract machine*, whose implementation is not a processor but an interpreter. Examples of abstract machines are the Java abstract machine and the .NET abstract machine.

A relatively recent kind of hybrid is called a *just-in-time* (jit) compiler, which is an interpreter with a kind of compiler available to it. While running a program, the interpreter keeps track of how often it performs each subprogram. When the number of calls to a particular subprogram becomes high enough to justify the cost, the interpreter compiles just that subprogram to machine language, and from then on runs the compiled version rather than using the interpreter for that subprogram. In practice, programs are found to spend a large majority of their time running a small number of subprograms, so this can greatly increase the execution speed of a program at a relatively small cost in compilation time.

The term "just in time" is derived from manufacturing practices in which materials and components do indeed arrive just in time to be used, reducing storage costs. For compilers, "just a little late" might be a more descriptive term, since a subprogram is compiled only after it has already been run in interpretive mode enough times to make it clear that it is worth compiling that subprogram. To compile every subprogram the first time it is called would slow down, not speed up, many executions. The term "on-the-fly compilation" is more descriptive.

7.6. Languages are not their implementations

Confusing a language with its implementation can lead you to incorrect conclusions. For example, to understand how efficient programs written in a given language tend to be, an obvious thing to do is to run them to find out. Should the efficiency of an implementation be used to characterize the efficiency of a programming language? Consider Lisp, an old language dating from the 1960s. Lisp was, for a long time, implemented exclusively via interpreters, which are relatively slow, so it was generally accepted that Lisp was a slow language. Today, most Lisp programs are run via compilers, and Lisp programs run very fast.

It comes as no surprise that different implementations exhibit different efficiencies. But what about meaning? To determine the meaning of a program or subprogram, you might just run it to see what it does, letting the language implementation serve as language definition. We might agree, for example, that whatever a particular implementation does is considered, by definition, to be correct. But what if there turns out to be a bug in the implementation? The bug cannot be fixed, or the implementation becomes, by definition, incorrect. Even worse, imagine that you need to learn how to use a language, and, as

documentation, someone hands you the listing of a compiler, and tells you to figure it out! Clearly, a language implementation is an awkward definition.

In spite of the obvious difficulties of using an implementation as a definition, programmers often find it convenient to use a language implementation as a definitive source about the language's meaning. For example, if you discover a new function, and you are not sure what that function does (probably due to poor documentation), you might find out by running some experiments. What you find out, of course, is what that particular implementation does. You might expect that all implementations would be required to do the same thing, but that is not always the case. Certain behaviors can be left unspecified in the definition of a language, and different implementations might do different things in those cases. For example, consider procedure test, written in Pascal, with comments written in braces.

```
procedure test;
  var i: integer;
  begin
    for i := 1 to 4 do begin end; {Do nothing}
    writeln(i) {Print i}
  end;
```

If you run this, something will be printed. But the definition of Pascal says that the value of a for-loop counter is unspecified at the end of the loop, so different implementations might print different values, and you can be misled by a test.[2]

7.7. Exercises

7.1. What are the main differences between a compiler and an interpreter?

7.2. Do all compilers translate to machine language?

7.3. Explain why interpreters tend to be more portable than compilers.

7.4. Explain what a linker does. Provide some details, not just a vague description.

7.5. What is the difference between a static linker and a dynamic linker?

7.6. Give two important reasons why an implementation of a programming language is not a good definition of the language.

7.7. What is a just-in-time compiler?

7.8. Suppose that you write an interpreter for a new language called Fruitcake. You write the interpreter in Lisp, with the intent of using a Lisp interpreter to run your Lisp program. Imagine that it takes 100 machine language instructions to perform one Lisp instruction, and it takes 200 Lisp instructions to perform one Fruitcake instruction. How many machine language instructions does it take to simulate one Fruitcake instruction?

[2] Typically, i will be either 4 or 5. But in Pascal, you can limit the type of variable i to be a value from 1 to 4, by declaring i to have *subtype* 1..4. If i has the value 5 after the loop, it would not even have the right type. Pascal allows a for-loop counter to have a value that is not merely unspecified, but is *undefined*, after the loop ends.

7.8. Bibliographic notes

The structure of compilers is described by Aho, Sethi and Ullman [5] and by Grune, Bal, Jacobs and Langendoen [48]. Aycock [8] gives a history of just-in-time compilers. Lindholm and Yellin [65] describe the Java abstract machine and byte code.

Chapter 8

Managing Memory and Subprogram Calls

8.1. Managing memory

As discusssed in Chapter 4, a programming language implementor needs to make choices about representation schemes for basic data structures, such as lists. But regardless of the representation scheme, which determines *how* to represent data, an implementation of a programming language needs to decide *where* in memory to store data, and how to determine when that data is no longer needed, so that the memory that it occupies can be reused. Three schemes for that are *static allocation*, *stack allocation* and *heap allocation*. Each of those allocation schemes has advantages and disadvantages.

8.1.1. Static allocation

The simplest method of managing memory is to place all data items in fixed memory locations, and never to move them. The memory occupied by each variable or constant is determined when the program is loaded into memory, and cannot change while the program runs. That method, called *static* allocation, was used by the first high level language, Fortran. It had the advantage of simplicity. Also, statically allocated variables were efficient to access, since direct addressing modes provided by processors could be used, and addresses did not need to be computed on the fly, which was expensive for early computers.

But with static allocation, the size of a program and the number and sizes of items stored in the memory cannot change while the program runs. That severely limits the kinds of data structures and algorithms that can be used. Arrays can be used, as long as it is clear exactly which arrays are needed, and how large they are, when the program is loaded into memory. Linked representation of lists, on the other hand, relies on the ability to create new list cells on the fly, and so cannot be used without some special mechanism for storing the list cells into statically allocated memory. Fortran is very successful for certain kinds of numerical computing that need speed and that rely mainly on simple data structures such as arrays, but it is not recommended for programs that need sophisticated data structures or that need to get more memory as the program runs.[1]

[1]Even early versions of Fortran did provide for a primitive way to get more memory, but all memory

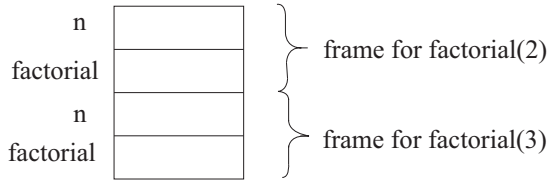
8.1.2. Dynamic allocation: the run-time stack

Early versions of Fortran did not allow the use of recursion, which requires two or more copies of a subprogram's variables to exist at once As more and more copies of a subprogram are created, more and more variables must also be created. Clearly, static allocation cannot be used for that.

Algol 60 does allow recursion, and that required its implementors to find a way to allocate memory on the fly. The solution that they used is to store variables in an area of memory called the *run-time stack*, or simply the stack for short. The stack is broken into *frames* of various sizes, each frame holding variables for one copy of a subprogram. (The instructions of the subprigrams are not themselves copied, since they never change.) Consider the following Algol 60 function, which computes factorials. It is called a procedure because all subprograms are called procedures in Algol 60. The result is stored into a variable that has the same name as the subprogram.

```
integer procedure factorial(n);
  integer n;
  begin
    if n = 0
      then factorial := 1
      else factorial := n * factorial(n-1)
  end
```

When computation of factorial(3) begins, a frame is allocated for it in the run-time stack, holding variables n and factorial. Then computation of factorial(2) is started, since its result is needed by factorial(3). A new frame is created for that copy of factorial, and the run-time stack looks like this, with the more recently created frame shown on the top.



Next factorial(1) is required, and then factorial(0). At its peak, the run-time stack holds four frames for copies of function factorial. When a subprogram is finished, it no longer needs its variables, so its frame can be removed from the stack. Frames are added to the top of the stack, and always deleted from the top.

Use of a stack for holding variables can be a considerable advantage over static allocation, even when recursion is not used. Suppose several subprograms are defined, each requiring an array of 1000 bytes. With static allocation, 1000 bytes must be given permanently to each subprogram, and those bytes will be allocated even when the subprogram is not running. With stack allocation, memory is only allocated for each array when the subprogram that needs that array starts running, and that memory is recovered for reuse as soon as the subprogram is done. So the run-time stack allows the memory for those arrays to be shared.

But there is another advantage that is sometimes even more important. With static allocation, a program is intimately tied to the variables that it uses. If you want to run

management was done by the program itself on what appeared to be a blank array of memory.

two copies of a program at once, you have no choice but to create two copies of both the program and the data. But when variables are stored in a stack, subprograms get their data from their own stack frames. To have two copies of a program running at once only requires that there be two copies of the data; the memory occupied by the programs themselves can be shared. For example, a large piece of software might be in use simultaneously by two users, or in two windows by one user. The operating system only needs to load one copy of its machine code into memory. Libraries similarly can be loaded only once, and shared by several concurrent programs. Programs or subprograms that can be run by several different programs at once are called *reentrant*.

Modern operating systems allow a given process to run several *threads* that are similar to several concurrently executing programs, but that share some memory, and that contribute to one overall goal. Different threads have separate run-time stacks, so that they do not interfere with one another. More and more, computers are being designed to take advantage of multithreading, and use of threads is constantly growing. Even modern versions of Fortran no longer rely on static allocation because of the advantages offered by the run-time stack.

8.1.3. Dynamic allocation: the heap

The use of a run-time stack allows a program to grow and shrink, but it has a fairly severe limitation. The memory allocated by a subprogram is always lost when that subprogram finishes, because the subprogram's frame is removed from the stack. It is impossible for a subprogram to create something that needs to be stored in newly allocated memory and to let that thing continue to exist when the subprogram returns to its caller. Using stack allocation, it is impossible to write a function that takes a list, such as $[2, 3, 4]$, and returns a longer list, such as $[1, 2, 3, 4]$, with another value added to the front, since the longer list must make use of newly allocated memory.

Heap allocation solves that problem. The *heap* is an area of memory that is initially unused by a program. It is managed by a heap memory manager, which can either be part of the library or part of the run-time support, depending on whether it is called explicitly or implicitly. A heap memory manager accepts requests to allocate memory, and *might* also accept requests to deallocate memory, so that the memory can be recycled and allocated again. An example of a heap memory manager is the one provided in the C standard library. To allocate a block of n bytes of memory, you call function malloc with n as a parameter. Expression malloc(n) allocates n bytes of memory from the heap and returns the address of the first of those n bytes. To recycle those bytes, you call procedure free, passing it the memory address given to you by malloc.

Managing the heap

Malloc not only needs to allocate memory, but it must keep track of which parts of memory are available to be allocated. Different versions of malloc behave a little differently, but all have some similarities, and we give a rough approximation of what malloc does.

One approach is to keep several *free space lists* of free blocks of memory. They are linked lists, where the pointers that link the lists together are stored in the memory blocks themselves. A list is kept for memory blocks of 16 bytes, another for blocks of 32 bytes, another for 64 bytes, etc., using higher and higher powers of two. A simple version of malloc always allocates a block whose size is a power of two, allocating 64 bytes when 40

are requested.

When malloc is called, it checks for a block in a free space list of an appropriate size. If it finds a block, it allocates that block, removing it from the front of the list. If it does not find a block of the appropriate size, it needs to get a block from a free space list with larger pieces. It takes that larger block and divides it in half. If no larger blocks are available, malloc asks for more memory from the operating system, adding one large block to one of its free space lists.

A call to malloc(n) actually allocates a little more than n bytes. For our simple version of malloc, a number of bytes m is selected such that $m \geq n + 4$ and m is a power of 2. Malloc places the number m in four bytes of memory just before the block whose address is returned. When a program calls free to return this memory to the heap manager, the parameter passed to free is the memory address that was returned by a call to malloc. Free looks in the memory immediately before that address, and finds the number m of bytes that were allocated. It then places this block at the front of the appropriate free space list.

Fragmentation

This simple memory allocator suffers from a problem. Occasionally, it must chop a large block into smaller blocks. If it never does the opposite operation, combining smaller blocks into larger blocks, then over time the average size of the available blocks tends to shrink. The memory allocator can end up in a state where it has a great deal of memory available, but almost all of it is in very small blocks. A request to allocate a large block might have to be denied, causing the program to run out of memory. This problem is called *fragmentation* of the memory, and is a serious one.

Fragmentation can be partially remedied by schemes that combine small blocks into larger blocks. For example, if free space lists are kept in ascending order by address, then it is easy to discover when two adjacent memory blocks are both in the same free space list. When they are, they can be combined, and moved into a list of larger blocks. That scheme somewhat alleviates, but does not fully solve, the problem of fragmentation. It is still possible to have a great deal of free space cut up into small blocks, with used space in between the free blocks, which makes combining adjacent free blocks impossible. Fortunately, fragmentation can be eliminated by a more sophisticated approach to memory management: garbage collection, discusses in the next section.

Difficulties with heap allocation

Suppose that a program mistakenly frees a block of memory that it still needs to use. What happens to the program? The potential problems are endless. The freed block will be placed into a free space list. Later, it might be allocated again. At that time, the same memory will be in use for two different purposes: the original purpose, for which it was not really finished, and the new purpose, for which it has just been allocated. The two uses interfere with one another, and chaos ensues. On the other hand, before the block is allocated again, it is both in use by the program and part of a free space list. If the program alters the memory block before it is allocated again, the change might disrupt the structure of a free space list (remember that the pointers that link the free space list are in the free memory), causing eventual chaos when the memory allocator attempts to use the damaged list. It is so difficult to predict what will happen if memory is freed prematurely that the only reasonable thing for a language definition to do is to say that a program enters a *semantic*

black hole, where all future behavior undefined.[2]

8.1.4. Garbage collection

Most modern programming language implementations avoid the memory deallocation problem altogether by not allowing a program to free memory that it has allocated. Instead, the system takes on the responsibility for identifying and freeing memory that is no longer in use. The part of the run-time support responsible for that is called a *garbage collector*. One problem faced by a garbage collector is determining where the unused memory is. There are two commonly used schemes for doing that.

Using reference counts

One scheme is to use *reference counts*. Each block of memory has some extra memory allocated with it, holding a count of the number of pointers currently held by the program that point to this block. Each time a new pointer to a block is created, the reference count of that block is increased, and each time a pointer is destroyed, the reference count is decreased. When the reference count becomes zero, the block is inaccessible, and it is moved to a free space list.

Reference counts are relatively straightforward to implement, but they suffer from some difficulties. One is that reference count management is *pervasive*. It must be done every time a pointer is created or destroyed. A program that makes frequent use of pointers will be slowed down by the need to manage reference counts. A more serious difficulty is that cyclic structures cannot be recovered. If two blocks contain pointers to one another, then their reference counts will remain positive, even when both blocks become inaccessible to the program. Finally there is the problem of the need to store the reference counts. Extra memory must be used, which increases the overall memory utilization of the program.

Mark/sweep garbage collection

An alternative approach is a *mark/sweep garbage collector*, which is run periodically, when available memory is low. Each block of data has an associated *mark bit*, which is used to indicate whether that block is in use. Initially, all of the mark bits are set to 0. The garbage collector begins with a *mark phase*, during which it scans through all memory that is accessible to the program, marking it all by setting the mark bits to 1 for the blocks that are encountered. To get going, the garbage collector marks all variables that are kept in the run-time stack. When a variable refers to a data structure via a pointer, that data structure is marked. The mark phase follows pointers, working its way through data structures. Of course, if a block of memory is encountered that has already been marked, it is not necessary to look inside that block again, or to follow any of the pointers that it contains.

After the mark phase, the garbage collector does a *sweep phase*, also called as *collect phase*, in which it scans the entire heap, freeing all blocks that have not been marked.

Mark/sweep garbage collectors have some advantages over other schemes. They can collect cyclic structures. There is no need to keep track of reference counts while the program runs. The program just runs as if it does not care whether unused memory is

[2] A careful definition needs to distinguish a *logical* block of memory, which is allocated and deallocated only once, from a *physical* block of memory, which might be allocated several times, each time holding another logical block of memory. The program enters a semantic black hole at the point where it makes use of a logical block of memory that has been deallocated.

ever deallocated. Finally, only one bit per block is needed for the marking, so memory requirements are low. A disadvantage is that the garbage collector needs to be able to recognize pointers (and to distinguish them from integers) so that it knows to follow them during the mark phase. Tags on data items will do the job.

Avoiding fragmentation

One of the big advantages of mark/sweep garbage collectors is that they can be designed to be very effective at avoiding fragmentation. A *relocating* garbage collector can move blocks to new locations, *compacting* them so that there is no unused memory between the used blocks. Then the unused portion is a single, contiguous block of memory. The biggest technical problem in performing the compactification is *relocation*: when a block of memory is moved, all pointers to it that are in other blocks must be changed to point to the new location. That can be done during the sweep phase, as long as information is available that tells the garbage collector how to change each pointer. The main issue is how the relocated pointers are found.

A simple relocating garbage collector keeps two heaps. At any given time, one of the heaps is in use and the other is fallow. Suppose that, at the start of a given garbage collection, heap A is in use and heap B is fallow. During the mark phase, the garbage collector copies each marked block from heap A to heap B, putting the blocks next to one another to eliminate holes. It leaves behind, in the old location of a block (in heap A), the address of its new location (in heap B). Think of it as like a forwarding address telling where each block now resides. At the end of the mark phase, the blocks in heap B contain the same bit patterns as they did when they were in heap A, since they are exact copies. Any pointers in those blocks are memory addresses in heap A. Each pointer in heap B needs to be changed to point the correct block in heap B. The garbage collector performs a *relocation* phase, during which it sweeps over heap B, replacing each pointer that points into heap A by the new location of the block to which the pointer refers. That new location is found in heap A, where it was put earlier. (It is the forwarding address.) Relocation is also done on the run-time stack, and all other places where pointers might be stored. Finally, heap A is marked fallow and heap B is marked in-use.

Of course, having two heaps, one of which is always not being used, doubles the memory requirements. Both time and memory can be saved by some simple improvements. Things stored in the heap tend to be of two kinds: long-lived things that are created early and stick around, and short-lived things that are created, used and abandoned quickly. A memory manager can keep a few blocks of memory for the heap. As values are moved during relocation (from one block to another), they are preferentially moved into lower numbered blocks. The result is that long-lived things tend to move into those blocks and stay there, leaving higher-numbered blocks for the short-lived things. During a given garbage collection, the lower numbered blocks might not be scanned at all under the presumption that almost all things in those blocks will still be accessible. Occasionally, the low numbered blocks need to be examined. But not every block needs to be compactified during every garbage collection; some can simply be left as is, so that only a few extra fallow block will suffice to manage relocation.

Impact of garbage collectors

Programs that need to free memory explicitly can have a significant fraction of their lines (and programmer time) devoted to determining when to free each item, making memory

management an unpleasant and difficult job for software developers. Worse than that, programs that make deallocation mistakes tend to be very difficult to debug; the consequences of the error often shows at a point far away from where it was made, and the error can consist of the program doing something that would appear impossible. (Remember that the program has entered a semantic black hole.) Languages whose implementations provide automatic garbage collectors tend to be much easier to use.

Lisp and Java are examples of languages all of whose implementations are equipped with garbage collectors. Typically, relocating mark/sweep garbage collectors are used. Most new languages require garbage-collecting implementations because of the enormous convenience that they provide.

8.2. Implementation of subprogram calls

8.2.1. Activations

The information in a frame of the run-time stack contains variables for a particular call of a subprogram, and also contains some additional information needed to manage subprogram calls and returns. As a whole, the information is called an *activation* of a subprogram. This section explores more detail about the information that is part of an activation and how a language implementation manages frames and the structure of the stack.

8.2.2. Pushing and popping frames

A program needs to be able to add and remove frames in the run-time stack and to locate the top frame, which belongs to the subprogram that is currently running. Since frames are of different sizes, to say that you know where a frame is in memory, you must know both its top and bottom boundaries. So the program evaluator keeps two pointers, the *stack pointer*, pointing to the top of the top frame, and the *frame pointer*, pointing to the bottom of the top frame.

Imagine that subprogram P calls another subprogram Q. A frame holding an activation of Q needs to be pushed on top of the stack, just above a frame holding an activation of P, as shown in Figure 8.1. As part of the calling process, P stores some information into the frame, such as the instruction counter and parameters, so that Q has enough information to start. After Q starts running, it enlarges the frame to make room for local variables, since only Q knows how many local variables it needs. Suppose that, after running for a while, Q is ready to return an answer to P. The evaluator must restore the stack pointer and the frame pointer to the top and bottom of the frame for P, so that P can continue. Setting the new stack pointer is easy: the top of P's frame is the same as the bottom of Q's frame, so it suffices to copy the frame pointer into the stack pointer. But how is the new value of the frame pointer, the bottom of P's frame, determined? A solution is to store the address of the bottom of P's frame at the bottom of Q's frame, in a variable called the *dynamic link*, and to copy that into the frame pointer. Figure 8.1 shows Q's frame holding a dynamic link pointing to the bottom of P's frame.

8.2.3. Scope and closures

Figure 8.2(a) shows a function definition, written in Cinnameg, where function g is written inside function f, and is allowed to use f's variables. The scope of x is the entire definition

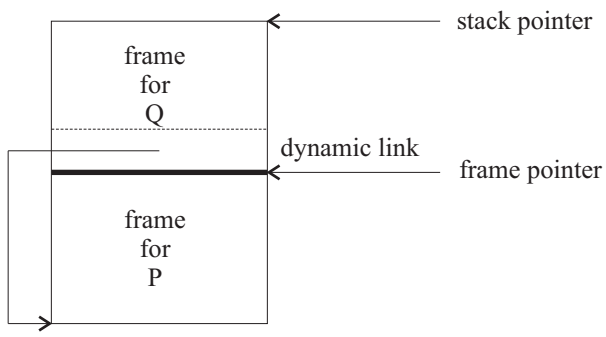


Figure 8.1: **Two frames on the run-time stack.** The stack pointer points to the top of the topmost frame, and the frame pointer points to the bottom of the topmost frame. The dynamic link in Q's frame points to the bottom of P's frame.

```
{f(x,z) = g(z) |
  {g(y) = x + y}
}
```

Figure 8.2: **Nested function definitions in Cinnameg.** The result returned by $f(x, z)$ is the value of expression $g(z)$ after defining function g as shown. Notice that g is allowed to use variable x that is in scope at the point where g is defined.

of function f. Since the definition of g lies within the definition of f, g can (and does) refer to x. But x is bound in the definition of f, not within g, so the value of x is stored in f's frame, not g's. Somehow, g needs to find f's frame. Notice that, when g is running, the stack frame for f will be immediately beneath g's in the run-time stack, since g is only called by f, so an obvious idea is to follow the dynamic link in g's frame to find the stack frame of f.

But that assumes that the only function that can call g is f, and that is not always the case. Figure 8.3 shows another example, a definition that contains two embedded definitions. Since blue calls green, which calls yellow, a frame for green will lie between the frames for yellow and blue. But yellow needs to find the value of x, which is stored in blue's frame. The dynamic link does not point to the frame where x is located.

```
{blue(x,z) = green(z) |
  {yellow(y) = x + y}
  {green(w) = yellow(w+1)}
}
```

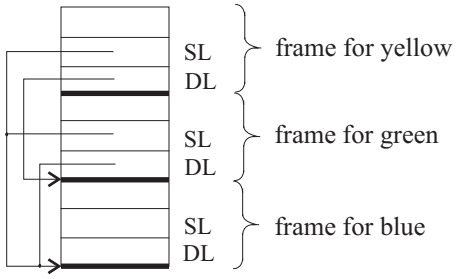
Figure 8.3: **Function blue, with functions yellow and green defined inside it.** The value of x is stored in blue's frame. Blue calls green, which calls yellow. To find the value of x, function yellow needs to skip over the intervening frame for green.

```
{case mapf(f, [])    = []
 case mapf(f, x::xs) = f(x) :: mapf(f, xs)
}
{h(x,z) = mapf(g,z) |
 {g(y) = x + y}
}
```

Figure 8.4: **Definition of mapf and a function h that uses mapf, in Cinnameg.** $h(3, [4, 5, 6, 7]) = [3+4, 3+5, 3+6, 3+7] = [7, 8, 9, 10]$. Function h calls mapf, which needs to call g. When mapf pushes a frame for g, it needs to know what to install as the static link in the new frame. It gets the pointer to install from the representation of function g as a function closure, as shown in Figure 8.5.

The definition of mapf is written in a style that is discussed in Chapter 9.

A solution to this problem is to add another link to each frame, called the *static link*, also called a *scope link* or *access link*. The static link in a frame for subprogram S points to the bottom of the frame where S was created, which is where S can find variables that were in scope when it was created. For example, since yellow is created inside blue, the static link in a frame for yellow will point to a frame for blue. When yellow is running, the run-time stack looks as follows, where SL is the static link and DL the dynamic link.



It is often the case, however, that solving one problem creates another, and this is no exception. Once you have decided to add a static link to each frame, you need to think about how you know what to store into it. When green calls yellow, for example, how does it know what to put into the static link of the new frame for yellow? It is tempting to let green pass its own static link on to yellow, but there is no reason to think that a subprogram can only call other subprograms that were created in the same place.

To illustrate what can happen, let's look at an example where a function is used far from the point where it is created. Some languages, including Cinnameg, allow you to pass a function as a parameter to another function. An example is function mapf(f, $[a, b, c]$) = $[f(a), f(b), f(c)]$. In general, mapf(f, x) computes f of each value in list x. Figure 8.4 shows a definition of mapf and of function h that uses mapf. When h runs, it will call mapf, which will need to call g (several times). But mapf and g might be defined in different modules, so mapf has no idea where g was created unless it is told.

The obvious thing to do is to tell mapf where g was created. What is passed to mapf, representing function g, is a pair called a *function closure* that contains (1) the instruction address where evaluation of g will start and (2) a pointer to the activation where g was

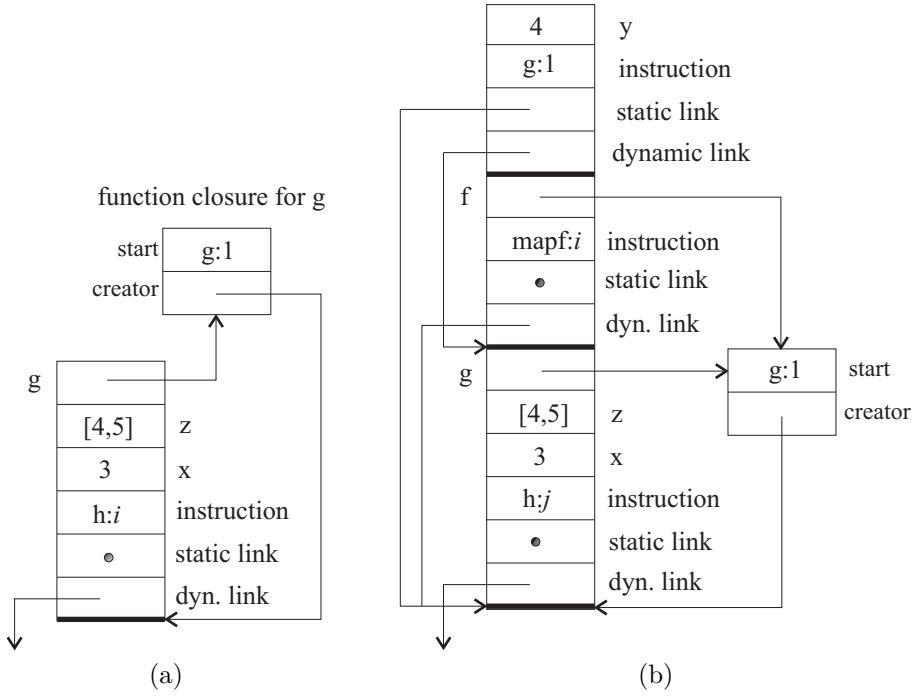


Figure 8.5: **Evaluation of** $h(3, [4, 5])$**.** Diagram (a) shows the frame for h in the run-time stack just before h calls mapf. Notice that the value of g in that frame is a pointer to a function closure. Diagram (b) shows the stack after h has called mapf and mapf has called g. The frame for mapf is abbreviated, showing only the value of f. Mapf gets the static link to use for g from the closure that represents g.

created. That pointer will become the static link when g is applied. For example, suppose that expression $h(3, [4, 5])$ is evaluated. When h defines g, it builds a function closure for it, with a static link pointing to the bottom of the current stack frame. (That is, h copies its own frame pointer into the new function closure.) Just before calling mapf, the activation of h is as shown in Figure 8.5(a). The value of g, stored in the activation of h, is a pointer to a function closure, and the closure points back to the activation where g was created, and also holds the instruction address where g will begin its computation. We write g:1 for the address where g starts, and h:i for some address in function h.

Now h calls mapf, and mapf calls $g(4)$, installing information from the function closure for g into a new frame, yielding the run-time stack shown in Figure 8.5(b). It is a simple matter for g to locate x by following the static link.

It is possible to define subprograms within subprograms within subprograms. To obtain a binding, it might be necessary to follow a static link to another activation, and then to follow that activation's static link yet again to find a binding. The compiler can see just how long a chain of static links must be followed to obtain a given binding, and can compile that information into the instructions that implement a subprogram.

8.2.4. Tail recursion and tail calls

As we have seen, computation of a function by recursion typically requires storing several activations of the function simultaneously in the run-time stack. But sometimes creating many frames can be avoided. Suppose that factorial is defined recursively in Cinnameg as follows, using a helper function factTimes$(n, x) = x * (n!)$.[3]

```
{case factTimes(0,x) = x
 case factTimes(n,x) = factTimes(n-1, n*x);
 factorial(n) = factTimes(n,1)
}
```

Start with factorial(3), which calls factTimes(3,1), which in turn calls factTimes(2,3). But instead of placing the frame for factTimes(2,3) above the frame for its caller, the frame for factTimes(2,3) can *replace* the frame for factTimes(3,1). The reason is that factTimes(3,1) wants to return exactly the result of factTimes(2,3) to factorial(3). There is no need for it to be around waiting to pass the result back to factorial; we can just ask factTimes(2,3) to give its result directly back to factorial(3).

Whenever the result of one function call is defined as exactly the same as the result of another, with no computation to be done with the result, we say that the function call is a *tail call*, and when the call is recursive, it is *tail recursive*. (In the case where a subprogram has no result, a tail call is one where nothing more is done after the call.) The altered stack management rule can be used for tail calls, with the new frame replacing the frame of the caller.

Recursive definitions that use tail recursion have the potential to be much more efficient than general recursive definitions, as long as a compiler or interpreter is prepared to recognize them, and to use the altered stack manipulation rule. But not all tail calls are recursive. For example, the definition of factorial does a tail call to factTimes, since it produces exactly the same result as factTimes produces. So the frame for factorial(3) can overwritten by the frame for factTimes(3,1).

Doing your own tail recursion improvement

Figure 8.6 shows an implementation of the Quicksort algorithm to sort an array, written in C++. There are two recursive calls to quicksort. The second call is tail recursive, since nothing is done after that call returns. It can be replaced by (1) an adjustment of the parameters (in this case, just changing low to mid+1), and (2) a jump back to the beginning of the function. Figure 8.7 shows a modified definition of quicksort that does the jump explicitly, using the goto feature of C++. You should see how the goto can be replaced by a while-loop. Try writing this version of quicksort without an explicit goto.

8.3. Exercises

8.1. Why can't Algol 60 implementations use static allocation like Fortran implementations?

[3]The style is that of Chapter 9.

```
int partition(int A[], int low, int high)
{
  int i,j;
  i = j = low + 1;
  while(j <= high) {
    if(A[j] < A[low]) {
      swap(A[i], A[j]);
      i = i + 1;
    }
    j = j + 1;
  }
  swap(A[i-1], A[low]);
  return i-1;
}

void quicksort(int A[], int low, int high)
{
  if(low < high) {
    int mid = partition(A, low, high);
    quicksort(A, low, mid-1);
    quicksort(A, mid+1, high);
  }
}
```

Figure 8.6: **Implementation of the Quicksort algorithm in C++.** We assume that statement swap(u, v) swaps the contents of variables u and v. The second recursive call to quicksort is tail recursive, but the first is not.

```
void quicksort(int A[], int low, int high)
{
start:
  if(low < high) {
    int mid = partition(A, low, high);
    quicksort(A, low, mid-1);
    low = mid+1;
    goto start;
  }
}
```

Figure 8.7: **A modified definition of quicksort that replaces the second recursive call by a change of the parameter (low) and a jump back to the start.** The effect is to modify the frame for quicksort to look like the frame for its recursive call. This is just the modification that an intelligent compiler can make for you.

8.2. Some Fortran programmers were unhappy with the introduction of recursion into Fortran. Why do you think they did not want recursion? Do you pay for it even if you do not use it?

8.3. What was the motivation for introduction of the heap?

8.4. What are the advantages of a relocating mark/sweep garbage collector over a garbage collector that uses reference counts?

8.5. What is one piece of software that probably needs to be able to free memory in the heap explicitly, rather than letting a garbage collector do that?

8.6. What is the main advantage of a garbage collector over letting a program manage its own memory?

8.7. What is fragmentation? Why is it a serious problem?

8.8. Our simple relocating garbage collector uses two heaps, one in use and the other fallow. That doubles the memory requirements of a program. What is the reason for keeping two heaps? Why not just (1) mark all accessible blocks, and (2) move all marked blocks to the beginning of the heap? Then only one heap is needed. Can you see any difficulties with that?

8.9. Language implementations that provide a heap and a garbage collector typically still use a run-time stack. Why do you think they don't just dispense with the stack and do all allocation in the heap?

8.10. What is the purpose of a dynamic link in the run-time stack?

8.11. When a new stack frame is pushed, how does the program know what to put in the new frame's dynamic link? How does it update the frame pointer and the stack pointer for the new frame?

8.12. When a subprogram returns to its caller, how are the frame pointer and stack pointer adjusted?

8.13. What is the purpose of the static link in a frame of the run-time stack?

8.14. What information is stored in a function closure?

8.15. When a function is called, how does the evaluator know what to put in the static link of the new frame?

8.16. In what ways is tail recursion more efficient than general recursion?

8.17. Using the definition of factorial that does not use factTimes, show the steps in evaluating factorial(3) using a run-time stack by showing a snapshot of the stack after each step. Perform steps leading up to a recursive call as a single step, to avoid too many snapshots of the stack.

8.18. Using the definition of factorial that uses factTimes, show the steps in evaluating factorial(3) using a run-time stack by showing a snapshot of the stack after each step. Perform tail calls by replacing stack frames. Perform steps leading up to a recursive call as a single step, to avoid too many snapshots of the stack.

8.19. It is easy to show that the first equation for factTimes is consistent with the assertion that factTimes$(n, x) = x * (n!)$.

$$\begin{array}{lcll} \text{factTimes}(0, x) & = & x & \text{from the definition of factTimes} \\ & = & x * 0! & \text{since } 0! = 1 \end{array}$$

Show that the second equation for factTimes is also consistent with the assertion that factTimes$(n, x) = x * (n!)$.

8.20. The definition of quicksort in Figure 8.7 uses a goto. Rewrite it to use a while-loop instead, avoiding use of a goto.

8.21. Quicksort needs to do two recursive calls, to sort two regions of the array, and it does not matter in which order they are done. You can keep the stack depth small by always performing the recursive call on the smaller region of the array, using the goto for the larger region. Modify the definition of quicksort so that it does the tail-recursive call on the larger region.

8.22. The definition of function *partition* in Figure 8.6 uses a loop. A loop can always be replaced by tail recursion. Show how to rewrite partition using tail recursion, without using a loop or using any kind of goto statement. (In a language that supports loops, that transformation does not make much sense. But the point is that loops can be simulated by tail recursion in languages that offer no direct support for loops.)

8.23. Show how to replace an arbitrary while-loop by tail recursion. You will need to create a helper function that represents the computation carried out by the loop body.

8.4. Bibliographic notes

Boehm [17] describes a garbage collector for C++ programs that allows C++ programmers to avoid freeing memory explicitly. Aho, Lam, Sethi and Ullman [5] provide extensive information on garbage collection techniques. The version of quicksort shown in Figure 8.6 is from Bentley [14].

Chapter 9

Equational Programming

9.1. Functions, equations and substitution

In mathematics, you typically define a function by an equation, such as

$$f(x) = 3(x) + 2. \tag{9.1.1}$$

You use a function by performing substitutions. For example, replacing x by 6 yields $f(6) = 3(6) + 2 = 20$. Already, we have the beginnings of programming, including function definition and computation. This chapter begins to explore how far that idea can take us as a form of *declarative programming.*.

9.2. Making decisions

Some functions can be defined by a single equation, but others, such as the function $\max(x, y)$ that produces the maximum of x and y, appear to need a mechanism for making decisions; you would expect a rule for computing $\max(x, y)$ to involve comparing x and y to decide which is larger. A first step toward making decisions is to allow expressions to produce boolean values, not just numbers. For example, expression $x > y$ yields either true or false. Then, to make decisions, you can use a *conditional* function $\text{cond}(a, b, c)$ that yields b if a is true, and yields c if a is false. For example, $\text{cond}(\text{true}, 4, 2) = 4$ and $\text{cond}(\text{false}, 4, 2) = 2$. So max is defined by equation $\max(x, y) = \text{cond}(x \geq y, x, y)$. A slightly more involved function $\text{sign}(x)$, defined to be -1 when x is negative, 0 when $x = 0$ and 1 when x is positive, is defined by the following equation.

$$\text{sign}(x) = \text{cond}(x > 0, 1, \text{cond}(x < 0, -1, 0)) \tag{9.2.1}$$

Figure 9.1 shows computation of sign(4) by doing repeated substitutions.

Using more than one equation

Although using a conditional function is all that is required to make decisions, an alternative is to write more than one equation, with a proviso indicating when each equation is true.

$$\begin{aligned}
&\text{sign}(4)\\
&\quad = \text{cond}(4 > 0, 1, \text{cond}(4 < 0, -1, 0)\\
&\quad = \text{cond}(\text{true}, 1, \text{cond}(\text{false}, -1, 0))\\
&\quad = \text{cond}(\text{true}, 1, 0)\\
&\quad = 1
\end{aligned}$$

Figure 9.1: **Evaluation of** sign(4) **by substitution, using Equation (9.2.1).** (Was it really necessary to evaluate $\text{cond}(4 < 0, -1, 0)$, or could it have been left as an unevaluated expression?)

Two equations suffice to define max.

$$\begin{array}{lcll}
\max(x, y) & = & x & \text{when } x \geq y\\
\max(x, y) & = & y & \text{when } x < y
\end{array}$$

To compute $\max(4, 6)$, you notice that only the second equation is true, since only its proviso is true, so that is the equation that you use. There can be any number of equations. For example, the sign function involves three equations.

$$\begin{array}{lcll}
\text{sign}(x) & = & 1 & \text{when } x > 0\\
\text{sign}(x) & = & 0 & \text{when } x = 0\\
\text{sign}(x) & = & -1 & \text{when } x < 0
\end{array}$$

9.3. Recursion

Some functions, such as $\text{factorial}(n) = n \cdot (n-1) \cdot \cdots \cdot 2 \cdot 1$, need more steps to evaluate as their arguments become larger. Equational programming handles that using recursion; a function can appear on the right-hand side of its defining equations. For example, notice that $\text{factorial}(n) = n \cdot \text{factorial}(n-1)$ for $n > 1$. In fact, if we define $\text{factorial}(0) = 1$, which is the usual rule, then $\text{factorial}(n) = n \cdot \text{factorial}(n-1)$ for all $n > 0$, leading to the following definition of factorial.

$$\begin{array}{lcll}
\text{factorial}(n) & = & 1 & \text{when } n = 0\\
\text{factorial}(n) & = & n \cdot \text{factorial}(n-1) & \text{when } n > 0
\end{array} \qquad (9.3.1)$$

Computing factorial(3) involves doing repeated substitutions, as shown in Figure 9.2.

9.4. Pattern matching

Look more carefully at evaluation of factorial(3). Equations (9.3.1) tell you how to compute factorial(n), but what you want is factorial(3). To make them the same, you solve equation $n = 3$ to determine a suitable value for n. It seems silly to consider this task as "solving" an equation, but looking at the substitution process as a matter of solving equations opens the possibility of slightly more involved equations. For example, suppose that you are told the following equation (with a proviso) about the factorial function.

$$\text{factorial}(n+1) = (n+1) \cdot \text{factorial}(n) \quad \text{when } n \geq 0 \qquad (9.4.1)$$

$$\begin{aligned}
&\text{factorial}(3)\\
&\quad = 3 \cdot \text{factorial}(3-1)\\
&\quad = 3 \cdot \text{factorial}(2)\\
&\quad = 3 \cdot (2 \cdot \text{factorial}(1))\\
&\quad = 3 \cdot (2 \cdot (1 \cdot \text{factorial}(0)))\\
&\quad = 3 \cdot (2 \cdot (1 \cdot 1))\\
&\quad = 6
\end{aligned}$$

Figure 9.2: **Evaluation of** factorial(3) **using Equations (9.3.1).** At each step, the leftmost expression is found that can be evaluated, and that does not contain any subexpressions that can be evaluated. For example, the equation for factorial is not used for factorial$(3-1)$ since it contains the smaller expression $3-1$, which can be evaluated to 2. In later steps, arithmetic simplifications are performed immediately for brevity.

Now to compute factorial(3), you want $n+1=3$, or $n=2$, and substituting $n=2$ into Equation (9.4.1) tells you that factorial(3) = $(2+1) *$ factorial(2). There is not much advantage here, but we will encounter situations shortly where this kind of thing significantly simplifies equations.

When an expression such as $n+1$ is used on the left-hand side of an equation, it is called a *pattern*. Patterns are allowed in equational programming when it is clear how to solve the simple equations that arise from using them, so that their solutions can be done automatically in the course of performing substitutions. The mechanisms for solving them must be part of the language implementation, or must be specified somehow in a program. We will assume that, at a minimum, the following equations can be solved for x and y, given a and b.

$$\begin{aligned}
x+1 &= a\\
(x,y) &= (a,b)
\end{aligned}$$

(The only solution to equation $(x,y)=(a,b)$ is $x=a$ and $y=b$.) Pattern forms can be mixed. For example, equation $\text{jump}(x,y+1) = x \cdot y$ can be used to evaluate jump(5,9) by solving pattern match equation $(x,y+1)=(5,9)$, with solution $x=5$ and $y=8$.

Another kind of pattern is a constant. For example, the basis case for factorial can be simplified to

$$\text{factorial}(0) = 1, \tag{9.4.2}$$

where the pattern is just the constant 0. To evaluate factorial(0), you solve equation $0=0$, which is trivially true. Since there are no variables, just use Equation (9.4.2) as is. Since equation $0=1$ has no solution you cannot use Equation (9.4.2) to compute factorial(1). In that case, select a different equation.

9.5. The form of an equational program

Up to now, we have introduced new ideas as they were needed. But it is important to settle on the rules of what an equational program looks like and how it is used in computation. An equational definition of a function f consists of a collection of one or more equations, each of the form

$$f(p) = r \text{ when } c$$

or

$$f(p) = r$$

where p is a pattern expression, c is a condition (an expression that produces a boolean value) and r is an expression that gives the answer. The pattern expression p must have a form such that an algorithm is available that solves equation $p = v$, where v is an abitrary given value. The algorithm either gives a value for each variable that occurs in p or says that no solution to equation $p = v$ exists.

9.6. The mechanics of substitution

Mathematically, equations are symmetric; saying $A = B$ is the same as saying $B = A$. But from the standpoint of computation, equations are not symmetric; you always replace the left-hand side of an equation by its right hand side, not the other way around. So, during computation, Equation (9.1.1) allows you to replace $f(x)$ by $3(x) + 2$, but never $3(x) + 2$ by $f(x)$. Other than that restriction, any kind of substitution can be done. Suppose, for example, that g is defined by $g(x, y) = x + x \cdot y$. Substituting (5+4) for x and $(9 + 2)$ for y tells you that $g((5 + 4), (9 + 2)) = (5 + 4) + (5 + 4) \cdot (9 + 2)$.

Be careful when you substitute expressions for variables. A naive substitution of $5+4$ for x and $9+2$ for y yields $g(5+4, 9+2) = 5+4+5+4 \cdot 9+2$. Because of precedence rules, that is not correct. In general, put parentheses around expressions that are substituted, unless the parentheses are not required to maintain the correct meaning.

To evaluate an expression, you perform substitution steps until a form is reached that does not need further evaluation. Each step proceeds as follows.

1. Select a subexpression of the form $f(e)$, where e can be any expression, for evaluation. How the selection is made is discussed in the next section under evaluation policies.

2. Select an equation $[f(p) = r$ when $c]$ or $[f(p) = r]$ that is part of the definition of f. How this choice is made is also a policy decision.

3. Solve pattern match equation $p = e$ for all of the variables in pattern p, treating expression e as a unit not to be changed. (That is, if e is not just a value, do not try to get its value here. If its value were desired, that would have been done earlier as a policy decision.) If the equation has no solution, then select a different equation, going back to step (2), since this one cannot be used. If there is a solution, proceed to the next step.

4. If there is a proviso [when c], substitute the solution found in Step 3 into expression c, yielding expression c'. Evaluate c'. If it yields false, then this equation cannot be used; select another equation. If c' is true, continue with the next step.

5. Substitute the solution found in step 3 into expression r, yielding modified expression r'. Replace $f(e)$ by r' to complete the substitution.

For example, suppose that function $s(x, y)$ is defined for integers (positive and negative) x and y by

$$\begin{array}{rcll} s(0, y) & = & y & \\ s(x, y) & = & s(-x, y) & \text{when } x < 0 \\ s(x + 1, y) & = & s(x, y + 1) & \text{when } x \geq 0 \end{array} \qquad (9.6.1)$$

and you need to evaluate s(1,7). An attempt to use the first equation requires solving pattern match equation $(0, y) = (1, 7)$, which has no solution. The second equation has a successful pattern match (solution: $x = 1$ and $y = 7$), but its proviso is false. The third equation has both a successful pattern match (solution: $x = 0$ and $y = 7$), and a true proviso. Using it yields replacement $s(1, 7) = s(0, 7 + 1)$. Continue doing substitutions on $s(0, 7 + 1)$.

Your mathematical experience encourages you to use some well-known facts about arithmetic operations, such as the associative law of addition, $x + (y + z) = (x + y) + z$. But when performing substitutions, all you have is a set of defining equations for each function, and the associative law is not one of those equations. Suppose that $r(n)$ is defined by

$$\begin{array}{lcll} r(0) & = & 0 & \\ r(n) & = & n + r(n-1) & \text{when } n > 0 \end{array}$$

Evaluation of $r(3)$ begins as follows.

$$\begin{array}{rcl} r(3) & = & 3 + r(3-1) \\ & = & 3 + r(2) \\ & = & 3 + (2 + r(2-1)) \end{array}$$

At this step, it is tempting to rearrange the parentheses and convert this to $5 + r(1)$. But that is not how evaluation is done. You must compute in the original order, as dictated by the parentheses.

9.7. Evaluation policies

When you go to evaluate an expression, there can be two or more substitutions available to you. For example, to evaluate $f(g(2)) + h(10)$, you could use an equation for g or for h or even for f. An evaluation policy determines which one to choose, usually with the goal of performing an efficient or simple evaluation.

A common evaluation policy is *left-to-right inside-out* evaluation, usually abbreviated to *inside-out evaluation*. At each step, an expression is chosen for replacement that does not contain any smaller subexpressions that could also be replaced. If there are several choices, the leftmost one is chosen. Figure 9.2 shows an inside-out evaluation. Inside-out evaluation usually works well, but sometimes it needs to be modified. For example, to evaluate $\text{cond}(a, b, c)$, it does not make sense to evaluate all of a, b and c right away. Instead, it suffices to evaluate a, and then to continue either by evaluating b (if a is true) or c (if a is false). Figure 9.1 shows an evaluation that can be shortened by that rule.

There are other evaluation policies that might be chosen. One is pure left-to-right evaluation, which selects the expression that begins at the leftmost place and that has an available rule. That rule evaluates $f(g(x))$ by selecting an equation for f rather than one for g. Evaluation policies are examined more in Section 10.5.

Choosing among equations

After a subexpression is chosen, an evaluation policy also needs to select an equation that can be used for it. A typical policy is to try the equations in the order in which they are written until one that can be used is found. But the equations can, in principle, be tried in any order; Section 9.10.2 has something to say about that.

Sometimes more than one equation gives an answer for a particular expression. For example, if the max function is defined by

$$\begin{aligned} \max(x, y) &= x \quad \text{when } x \geq y \\ \max(x, y) &= y \quad \text{when } y \geq x \end{aligned}$$

then both equations can be used to compute max(4,4). There is nothing wrong with that, as long as both equations give the same answer. For example, if the equations are tried in the order written, the first one will be used.

9.8. Equational programming with lists

Up to now we have only defined functions on numbers. But equational programming shows its real power when used with data structures such as lists and trees, where functions often have short and simple definitions that tend to work without much debugging. This section illustrates functions involving lists.

We need some basic operations on lists to get going. It suffices to have one constant, $[\,]$, standing for an empty list, and four functions: head(x) is the head of list x (its first member); tail(x) is the tail of list x (all but its first member); null(x) is true if $x = [\,]$, and false otherwise; and $h :: t$ is the list whose head is h and whose tail is t. The empty list does not have a head or a tail, so both head($[\,]$) and tail($[\,]$) are undefined. We will write $x = [\,]$ in programs to mean null(x). But we only use this to test whether a list is empty. A general list equality test is not one of the basic functions.

The :: operator needs some explanation. The definition is that $h :: t$ is the list whose head is h and whose tail is t. Since head($[1, 2, 3, 4]$) $= 1$ and tail($[1, 2, 3, 4]$) $= [2, 3, 4]$, it must be the case that $1 :: [2, 3, 4] = [1, 2, 3, 4]$. The :: operator adds one value to the beginning of a list. Do not expect it to do more than that. It does not add a value to the end of a list, and it does not concatenate lists. For example, $[1, 2, 3] :: [4, 5, 6] = [[1, 2, 3], 4, 5, 6]$, not $[1, 2, 3, 4, 5, 6]$. By convention, operator :: associates to the right, which means that $a :: b :: c$ is understood to mean $a :: (b :: c)$. So $5 :: 3 :: [\,] = 5 :: [3] = [5, 3]$. Notice the empty list at the end of $5 :: 3 :: [\,]$; it is necessary. Expression $a :: b$ is not the same as $[a, b]$.

If lists are represented in linked form, then each of the four operations (null, head, tail and ::) takes constant time to perform. For example, computing $h :: t$ just requires building one list cell, and computing tail(x) just involves getting the "next" pointer.

Computing the length of a list

Define length(x) to be the number of members in list x. For example, length($[5, 6, 7]$) $= 3$ and length($[\,]$) $= 0$. Notice that the length of a nonempty list is one greater than the length of its tail, suggesting the following definition of length.

$$\begin{aligned} \text{length}(x) &= 0 && \text{when } x = [\,] \\ \text{length}(x) &= 1 + \text{length}(\text{tail}(x)) && \text{when } x \neq [\,] \end{aligned} \tag{9.8.1}$$

Pattern matching with lists

Pattern matching equations that involve the :: operator are straightforward to solve. For example, equation $h :: t = [2,\ 5,\ 7]$ has exactly one solution, $h = 2$ and $t = [5, 7]$. In general,

equation $h :: t = v$ has solution $h = \text{head}(v)$ and $t = \text{tail}(v)$. Equation $h :: t = [\,]$ has no solution, so pattern $h :: t$ can only match a nonempty list. Using patterns yields a simpler definition of length.

$$\begin{array}{rcl} \text{length}([\,]) & = & 0 \\ \text{length}(x :: xs) & = & 1 + \text{length}(xs) \end{array} \tag{9.8.2}$$

Note that the provisos are no longer needed; pattern $x :: xs$ can only match a nonempty list.

Is a given list in ascending order?

Suppose that ascend(xs) is supposed to be true if list xs is in ascending order. For example, ascend($[2, 4, 6, 7]$) = true, but ascend($[2, 4, 3, 5]$) = false. For long lists, you check whether the order is ascending by checking whether the first two members are in the correct order and then checking whether the tail of the list is also in ascending order. But that only makes sense if the list has at least two members. Smaller lists need to be handled as special cases.

$$\begin{array}{rcl} \text{ascend}([\,]) & = & \text{true} \\ \text{ascend}([a]) & = & \text{true} \\ \text{ascend}(a :: b :: r) & = & a < b \text{ and } \text{ascend}(b :: r) \end{array} \tag{9.8.3}$$

Membership test

Now let's define function member(x, ys), which yields true if x is a member of list ys. For instance, member(3, $[2, 5, 3, 6]$) = true, but member(3, $[2, 5, 7]$) = false. The empty list has no members, so member(x, $[\,]$) is certainly false. Evidently, x is in $y :: ys$ if either $x = y$ or x is a member of ys, leading to the following definition of member.

$$\begin{array}{rcl} \text{member}(x, [\,]) & = & \text{false} \\ \text{member}(x, y :: ys) & = & x = y \;\text{ or }\; \text{member}(x, ys) \end{array} \tag{9.8.4}$$

For efficiency, you would not want to continue looking for x after you have already found it. By convention, boolean operations **and** and **or** are *short-circuited*, meaning that (x and y) abbreviates cond($x, y,$ false) and (x or y) abbreviates cond($x,$ true, y). Using that convention, the recursive call member(x,ys) will not be evaluated when test $x = y$ is true.

Taking a prefix of a list

Suppose that you want to extract the first n members of a given list. Say that take(3, $[2, 4, 6, 8, 10]$) = $[2, 4, 6]$, and, in general, take(n, x) yields the length n prefix of list x. There is a special case where take is asked to take more than is available; in that case, by convention, say that it takes the entire list. For example, take(20, $[200, 400]$) = $[200, 400]$. Equations for take are straightforward to derive.

$$\begin{array}{rcll} \text{take}(0,\, xs) & = & [\,] & \\ \text{take}(n,\, [\,]) & = & [\,] & \\ \text{take}(n,\, x :: xs) & = & x :: \text{take}(n - 1,\, xs) & \text{when } n > 0 \end{array} \tag{9.8.5}$$

The last equation uses the :: operator to build the list. Notice that take(3, [a, b, c, d]) = [a, b, c] = $a :: [b,\, c] = a :: \text{take}(2,\, [b,\, c,\, d])$. Figure 9.3 shows an inside-out evaluation of take(2, [a, b, c, d]).

$$\begin{aligned}
&\text{take}(2,\ [a,\ b,\ c,\ d]) \\
&\quad = a :: \text{take}(1,\ [b,\ c,\ d]) \\
&\quad = a :: b :: \text{take}(0,\ [c,\ d]) \\
&\quad = a :: b :: [\,] \\
&\quad = a :: [b] \\
&\quad = [a,\ b]
\end{aligned}$$

Figure 9.3: **An inside-out evaluation of expression take(2, [a, b, c, d]) using Equations (9.8.5).**

$$\begin{aligned}
&\text{cat}([2,\ 3,\ 4],\ [5,\ 6]) \\
&\quad = 2 :: \text{cat}([3,\ 4],\ [5,\ 6]) \\
&\quad = 2 :: 3 :: \text{cat}([4],\ [5,\ 6]) \\
&\quad = 2 :: 3 :: 4 :: \text{cat}([\,],\ [5,\ 6]) \\
&\quad = 2 :: 3 :: 4 :: [5,\ 6] \\
&\quad = 2 :: 3 :: [4,\ 5,\ 6] \\
&\quad = 2 :: [3,\ 4,\ 5,\ 6] \\
&\quad = [2,\ 3,\ 4,\ 5,\ 6]
\end{aligned}$$

Figure 9.4: **Inside-out evaluation of cat([2, 3, 4], [5, 6]) using Equations (9.8.6).** Be careful to remember that :: associates to the right, so evaluation must be done in the order shown. (It would not make sense to try to evaluate $2 :: 3$, since that is a meaningless expression).

Concatenating lists

Suppose that you want to concatenate, or join, two lists. Say that $\text{cat}([2, 3],\ [4, 2]) = [2, 3, 4, 2]$ and $\text{cat}([7, 4, 3],\ [8, 9, 10]) = [7, 4, 3, 8, 9, 10]$. Notice that $\text{cat}([2, 3, 4],\ [6, 5, 2]) = 2 :: 3 :: 4 :: [6, 5, 2]$, and

$$\begin{aligned}
\text{cat}([a,\ b,\ c,\ d], y) &= a :: b :: c :: d :: y \\
&= a :: (b :: c :: d :: y) \\
&= a :: \text{cat}([b,\ c,\ d], y).
\end{aligned}$$

In general, $\text{cat}(x :: xs,\ ys) = x :: \text{cat}(xs, ys)$. That handles the case where the first parameter is not an empty list. But the case of an empty first parameter is obvious, leading to Equations (9.8.6).

$$\begin{aligned}
\text{cat}([\,], ys) &= ys \\
\text{cat}(x :: xs, ys) &= x :: \text{cat}(xs, ys)
\end{aligned} \tag{9.8.6}$$

Figure 9.4 shows an inside-out evaluation of expression $\text{cat}([2, 3, 4],\ [5, 6])$.

9.9. Loops and recursion

Equational programming is surprisingly versatile, and it tends to yield short, simple and reliable function definitions. As a result, it can be an economical tool, since it allows you to move on to other tasks. But the equational style is an acquired taste. This section compares

and contrasts equational programming with imperative programming, not only to see where the differences lie, but to help understand how to modify imperative ideas so that they work in an equational setting.

If you are accustomed to working with data structures such as linked lists in an imperative or object-oriented programming language, your natural tendency will be to think in terms of modifying a data structure. Take, for example, the problem of concatenating two lists. If you think in terms of a linked representation of lists, you imagine two lists $[1, 2]$ and $[3, 4]$ represented as follows.

You will be tempted to perform the concatenation by changing the null link in the list node that contains 2 so that it points to the list node that contains 3. There are two serious problems with that for the setting of equational programming. One is that it changes what is stored in a list node. That is, it treats the list node as a *variable*. In equational programming, there are no variables. You compute new values from old ones without modifying the old values. A second problem is that no operation has been provided for modifying a list. Only the constant $[\,]$ and the four functions null, ::, head and tail are available. The list operations are pure functions; they do not call for changing anything.

As discussed in Section 4.9, data structures that support operations that modify the data are called *ephemeral* or *mutable* data structures because information is lost when an update is performed. A data structure that has only pure functions defined on it is called a *persistent* or *immutable*. data structure, since anything that is created persists, and cannot be destroyed, except when it becomes inaccessible to the program and can be recycled by a garbage collector. Equational programming uses only persistent data structures. For example, Equations (9.8.6) define concatenation as as a function, computing the new list from the old ones.

Example: Sorting a list

A sorting function takes a list and reorders it into ascending (or, more generally, nondescending) order. Of course, in an equational program, you cannot actually change the list; instead, a sorting function produces a new list. For example, $\text{sort}([5, 2, 7, 1]) = [1, 2, 5, 7]$.

Sorting an empty list is easy: $\text{sort}([\,]) = [\,]$. So all that is left is to see how to sort a nonempty list; an example helps to see what $\text{sort}(x :: xs)$ is. To sort list $[6, 7, 5, 4, 3]$, first sort the tail $[7, 5, 4, 3]$, yielding $[3, 4, 5, 7]$. Now insert 6 where it goes, yielding $[3, 4, 5, 6, 7]$. The insertion can be done by another function, insert, were $\text{insert}(6,\ [3, 4, 5, 7]) = [3, 4, 5, 6, 7]$. More generally, $\text{insert}(x,\ L)$ produces a sorted version of $x :: L$, but requires that L is already sorted. That suggests the following definition of sort, which implements an algorithm called *insertion sort*.

$$\begin{aligned} \text{sort}([\,]) &= [\,] \\ \text{sort}(x :: xs) &= \text{insert}(x,\ \text{sort}(xs)) \end{aligned}$$

Three cases suffice to define $\text{insert}(x,\ L)$, one for $L = [\,]$, one for the case where x goes at the

beginning of the result list, and one for the case where x does not belong at the beginning.

$$\begin{array}{rcll} \text{insert}(x,\ [\,]) & = & [x] & \\ \text{insert}(x,\ y :: ys) & = & x :: y :: ys & \text{when } x \leq y \\ \text{insert}(x,\ y :: ys) & = & y :: \text{insert}(x,\ ys) & \text{when } x > y \end{array}$$

Example: List reversal

Suppose that you would like to define function reverse(x) to yield the reversal of list x, so that reverse([2,5,3,6]) = [6,3,5,2]. A fairly simple recursive definition, called *naive reverse*, is as follows. For convenience of notation, we use operator ++ to indicate concatenation of lists (function cat from Chapter 9).

$$\begin{array}{rcl} \text{reverse}([\,]) & = & [\,] \\ \text{reverse}(x :: xs) & = & \text{reverse}(xs) \mathbin{+\!\!+} [x] \end{array} \tag{9.9.1}$$

Notice that reverse([a, b, c]) = [c, b, a] = [c, b] ++ [a] = reverse([b, c]) ++ [a]. The second equation above generalizes that observation.

Unfortunately, naive reverse can be very time-consuming (justifying its name). Exercise 9.6 asks you to show that concatenating two linked lists using function cat takes time proportional to the length of the first list. Reversing $[a, b, c, d]$ requires doing concatenations [d] ++ [c], [d, c] ++ [b] and [d, c, b] ++ [a]. In general, the total amount of work used to reverse a list of length n is proportional to $1 + 2 + \cdots (n-1) = n(n-1)/2$.

There is no reason to believe that reversing a list should be so expensive, and it is not, if you just adopt a better approach. Think about reversing a deck of cards. You start by holding the deck in your hand. You remove the top card, and place it in a new stack on a table. Then you successively remove cards from the original deck in your hand and place them on top of the stack on the table, until eventually you have the entire deck on the table, in reverse order. That algorithm is naturally expressed as a loop; the following loop sets variable *answer* to the reversal of list x.

```
y = x
r = [ ]
while y ≠ [ ] do
  r = (head y) ::  r
  y = tail y
end while
answer = r
```

Typically, a loop has some *loop control variables* that change as the loop runs. Our reversal loop has two loop control variables, y and r, where you can think of y as the deck of cards in your hand and r the deck on the table. Look at the values that y and r take on each time the top of the loop is reached, for the example where $x = [1, 2, 3, 4]$.

y	r
$[1, 2, 3, 4]$	$[\,]$
$[2, 3, 4]$	$[1]$
$[3, 4]$	$[2, 1]$
$[4]$	$[3, 2, 1]$
$[\,]$	$[4, 3, 2, 1]$

A *loop invariant*[1] is a property of the loop control variables (and possibly some other variables that do not change during the loop) that is true each time the loop reaches its top, and a well-chosen loop invariant can help a programmer understand what a loop is accomplishing. It expresses an unchanging aspect of a process that is constantly changing variables. In terms of the table for the reversal loop, a loop invariant is an assertion about y and r that is true for every line of the table. A useful loop invariant is that reverse(x) = reverse(y) ++ r. That is, the reversal of y followed by r yields the reversal of the original list x, at every line of the table. The line where $y = [3, 4]$ and $r = [2, 1]$ satisfies the loop invariant because reverse([1,2,3,4]) = reverse([3,4]) ++ [2,1].

When converting a loop to a recursive definition, there is a concept related to a loop invariant called an *invariant function*. If the control variables are y and r, then an invariant function is a function inv(y, r) that produces the same value at every line of the table. For reverse, you should discover that inv($[1, 2, 3, 4], [\,]$) = inv($[2, 3, 4], [1]$) = inv($[3, 4], [1, 2]$), etc. A function that has the desired property is *shunt*, defined by shunt(y, r) = reverse(y) ++ r. At every line in the list reversal loop table, shunt(y, r) yields the same answer, reverse(x).

To convert a loop to a recursive function definition, write a definition of an invariant function for the loop. The loop offers advice on how to define its invariant function. Suppose that the loop table has two consecutive lines,

y	r
y'	r'

Then, clearly, the invariant function definition should be written so that inv(y, r) = inv(y', r'). For example, shunt($[1, 2, 3, 4], [\,]$) = shunt($[2, 3, 4], [1]$) = shunt($[3, 4], [2, 1]$) = shunt($[4], [3, 2, 1]$) = shunt($[\,], [4, 3, 2, 1]$). In general,

$$\text{shunt}(x :: xs, r) = \text{shunt}(xs, x :: r). \tag{9.9.2}$$

Another equation handles the case of an empty list.

$$\text{shunt}([\,], r) = r \tag{9.9.3}$$

That equation corresponds to the step at the end of the loop that computes the answer from the control variables. (Remember that we made answer = r in the loop.) You should be able to check not only that these equation describe what is happening in the loop, but also that they are correct according to the definition shunt(y, r) = reverse(y) ++ r.

Loops usually also have some initialization. For example, the reverse loop initializes $y = x$ and $r = [\,]$. That is taken into account by adding another function that calls shunt with specific parameters. The reverse function is defined as follows, in terms of shunt.

$$\text{reverse}(x) = \text{shunt}(x, [\,]). \tag{9.9.4}$$

Evaluation of reverse([2,5,3,6]) by this new definition of reverse is shown in Figure 9.5. If you look carefully, you can see the playing card method being performed. The first of the two lists is what is in your hand, and the second is what is on the table. The algorithm is played out in the evaluation, and so is implicit in the equational definition.

The number of steps to reverse a list of length n by this method is $n + 2$, which is certainly much better than the $n(n - 1)/2$ steps required by naive reverse. Another thing to notice is that, in the new reversal algorithm, the size of the expression being evaluated stays the same throughout the computation. The reason is that the definition of shunt is tail recursive (see Chapter 8), and that can lead to a substantial saving in memory.

[1]Loop invariants are discussed in more detail in Chapter 24.

$$\begin{aligned}
&\text{reverse}([2,5,3,6])\\
&\quad = \text{shunt}([2,5,3,6],[\,])\\
&\quad = \text{shunt}([5,3,6],[2])\\
&\quad = \text{shunt}([3,6],[5,2])\\
&\quad = \text{shunt}([6],[3,5,2])\\
&\quad = \text{shunt}([\,],[6,3,5,2])\\
&\quad = [6,3,5,2]
\end{aligned}$$

Figure 9.5: **Evaluation of a list reversal using a more efficient definition of reverse, Equations (9.9.2), (9.9.3) and (9.9.4).**

9.10. Equational programming in Cinnameg

Although you can translate a simple equational definition into just about any programming language that supports recursion, it works best in a language that supports equations directly. Here, we see how to convert equations to Cinnameg with very little modification from their mathematical forms, making equations a viable and practical way of defining functions.

To apply a function in Cinnameg, just write the function followed by its argument. Parentheses are not required. For example, expression `sqrt 4.0` evaluates to 2.0.

Feel free to write parentheses for clarity. Function application has higher precedence than binary operators, so $f(x + y)$ is not the same as $fx + y$.

To express a definition with one equation in Cinnameg, wrap the equation in braces, as in the following definition of function sqr, where $\text{sqr}(x) = x^2$.

```
{sqr(x) = x * x}
```

The max function can be written with one equation or two. When you use more than one equation, introduce each equation with the word **case**. Figure 9.6 shows both approaches. Cinnameg supports lists using the same notation that we have been using. Figure 9.7 shows a definition of cat.

9.10.1. Writing examples

Cinnameg allows you to write examples with a function definition. Examples serve three purposes. They augment the documentation, so that a human reader can understand what the function does. They let the compiler know something about the types that you intend a function's parameter and result to have. And they help to check that a function definition is at least partially working; examples are run when a program starts, and any that fail to work are reported. The following defines sqr with two examples.

```
{sqr(x) = x * x}
Example sqr(3) = 9.
Example sqr(5) = 25.
```

9.10.2. Search order

When you define a function by more than one equation in Cinnameg, the normal evaluation policy is to try the equations in the order in which they are written until one that succeeds

```
{max(x,y) =
  If* x >= y
    then x
    else y
  %If*
}
```

(a)

```
{case max(x,y) = x when x >= y
 case max(x,y) = y when y >= x
}
```

(b)

Figure 9.6: **Two definitions of the max function written in Cinnameg.** The conditional expression cond(a, b, c) is written `If*` a `then` b `else` c `%If*` in Cinnameg. Generally, when a capitalized word begins an *expression*, it has an asterisk to emphasize that. There is also an If-statement with no asterisk after If.
To use more than one equation, write **case** in front of each equation. Notice that there is no period at the end of a case. A period stands for an end marker, and only words that begin with upper case letters have end markers.

```
{case cat( [], ys)    = ys
 case cat(x::xs, ys) = x :: cat(xs, ys)
}
```

Figure 9.7: **A definition of the cat function in Cinnameg.** Compare to Equations (9.8.6). You do not really need this function since concatenation is already available in Cinnameg as operator ++. For example, $[2, 4, 6]$ ++ $[8, 10] = [2, 4, 6, 8, 10]$.

is found. Only the first one that works will be used, so you can presume that each equation will only be tried when all of the preceding equations have failed. To emphasize that to a reader, write the word **first** just before the first case. The following definition of max illustrates.

```
{first
 case max(x,y) = x when x >= y
 case max(x,y) = y
}
```

Of course, the second equation is only true when $x \leq y$, but its proviso can be omitted since it will only be tried in the case where $x < y$. That makes a more efficient definition, since the second proviso does not need to be checked. But it has an unpleasant characteristic as well. Since the second equation relies on an *implicit proviso* that the preceding case does not apply, you can only check it by first understanding the prior case. If there are several equations, checking each one can be difficult. Sometimes, you prefer to write a definition so that each case can be checked by itself, independently of the other cases. To do that in Cinnameg, write the word **one** just before the first case, indicating that the cases can be tried in an arbitrary order, not necessarily in the order written. Each case must stand on its own. The definition of function max can be written as follows.

```
{one
 case max(x,y) = x when x >= y
 case max(x,y) = y when y >= x
}
```

9.10.3. Giving things names

Sometimes you want to give names to intermediate results in computations either because you intend to use those values more than once (and only want to compute them once) or because you find that the names make the program more readable. To illustrate, function $\text{pow}(x, n) = x^n$ can be defined, for a positive integer power n, by three equations.

$$\begin{array}{rcl} \text{pow}(x, 1) & = & x \\ \text{pow}(x, 2 * n) & = & \text{pow}(x, n) * \text{pow}(x, n) \\ \text{pow}(x, n + 1) & = & x \cdot \text{pow}(x, n) \end{array} \tag{9.10.1}$$

The second equation uses pattern 2*n, which, since n is an integer, can only match an even integer. Notice that the second equation uses $\text{pow}(x, n)$ twice. Since the value will be the same both times, it makes sense to compute it just once, and to give it a name. Figure 9.8 shows a definition of that does that.

9.10.4. List notations

Programming languages make some jobs easier by providing convenient notations for common tasks. This section describes some notation for lists that are available in Cinnameg.

Counting lists

Notation `[m, ..., n]` stands for the list $[m, m + 1, \ldots, n]$. For example, $[1, \ldots, 5]$ is the list $[1, 2, 3, 4, 5]$. If $y < x$ then $[x, \ldots, y]$ produces $[\,]$. Notation `[1,3, ..., 9]` stands for list [1,3,5,7,9].

```
{case pow(x, 1)     = x
 case pow(x, 2*n)   = p*p |
                      {p = pow(x, n)}
 case pow(x, n+1)   = x*pow(x, n)
}
```

Figure 9.8: **Definition of pow in Cinnameg illustrating naming.** A left brace introduces a variable definition. Expression E `|` S calls for performing S, in this case defining p, and then producing the value of expression E. Pattern $2 * n$ only matches an even number, and solving equation $2 * n = k$ makes $n = k/2$.

List comprehensions

Expression `[x*x | x from [1, ..., 5]]` stands for list $[1, 4, 9, 16, 25]$, the list of squares of the members if list $[1, \ldots, 5]$. More generally, a *list comprehension* describes a list by giving a source list ([1,... ,5] in the example) and an expression telling what is put in the result list (x*x in the example). A list comprehension can also contain a *filter*, a condition indicating which members of the source list to look at. (Only those that make the condition true are considered.) For example, odd?(n) is true if n is odd and `[x*x | x from [1, ..., 5], odd?(x)]` evaluates to list $[1, 9, 25]$. Similarly, `[n | n from [2, ..., 20], prime?(n)]` produces the list $[2, 3, 5, 7, 11, 13, 17, 19]$ of all prime numbers between 2 and 20.

List comprehensions can make some algorithms easy to express. Suppose that a function listSum has already been defined, where listSum(x) is the sum of the numbers in list x. Then the sum $\sum_{k=1}^{n} f(k)$, for some function f, can be expressed easily and clearly using a list comprehension and the sum function.

```
listSum [f(k) | k from [1, ..., n]]
```

List comprehensions make it easy to express the Quicksort algorithm for sorting a list. Quicksort says to sort a nonempty list L by using the head of L as a *pivot*. Separate the tail of L into two lists A and B, where A consists of all members of tail(L) that are less than or equal to the pivot and B consists of all members of tail(L) that are greater than the pivot. Then sort lists A and B, using Quicksort, yielding sorted lists A' and B'. Finally, create the result list by concatenating A', the pivot and B'. For example, to sort list $L = [6, 3, 8, 9, 2, 4, 7]$, the pivot is 6, $A = [3, 2, 4]$ and $B = [8, 9, 7]$, $A' = [2, 3, 4]$ and $B' = [7, 8, 9]$. Concatenating A', the pivot and B' yields [2,3,4,6,7,8,9]. Figure 9.9 defines Quicksort using list comprehensions to get the two lists A and B.

9.10.5. Running a program

Although you can do testing with examples, you will want to use your functions in a complete program. Figure 9.10 shows a complete module that tests the insertion sort function described in Section 9.9.

9.11. Exercises

9.1. What is the value of each of the following expressions? Read them carefully.

```
{case quicksort [] = []
 case quicksort (pivot :: rest) =
    quicksort(smaller) ++ [pivot] ++ quicksort(larger) |
      smaller = [x | x from rest, x <= pivot]
      larger  = [x | x from rest, x > pivot]
 }
```

Figure 9.9: **Quicksort using list comprehensions.** Operator ++ concatenates lists. Compare this to an imperative definition of Quicksort, such as the one in Figure 8.6.

```
Module tester
  {case insert(x, [])    = [x]
   case insert(x, y::ys) = x::y::ys        when x <= y
   case insert(x, y::ys) = y::insert(x, ys)
  }
  Example insert(5,[2,4,6,8]) = [2,4,5,6,8].
  {case sort([])    = []
   case sort(x::xs) = insert(x, sort(xs))
  }
  Example sort([5,2,4,7,1]) = [1,2,4,5,7].
  Execute
   Displayln(sort([3,21,8,2,9,15])).
   Displayln(sort("word")).
  %Execute
%Module
```

Figure 9.10: **A complete module that defines and tests a sorting function.** The module is named tester. The Execute part steps aside from functional programming. It contains statements to be performed, in this case displaying the results of sorting two given lists.

(a) $2 :: [\,]$

(b) $3 :: [4,7]$

(c) $[3 :: [4,7]]$

(d) $2 :: 4 :: 5 :: [\,]$

(e) head([$2 :: [3]$, $4 :: [5]$])

(f) tail([3])

(g) tail([4,5])

9.2. Show an inside-out evaluation of length($[1, 2, 3, 4]$), using Equations (9.8.1). Be careful not to make use of the associative law of addition.

9.3. Using Equations (9.8.5), show an inside-out evaluation of take(2, $[4, 3, 2]$).

9.4. Using Equations (9.8.5), show an inside-out evaluation of take(5, $[4, 3, 2]$). Why was no special case needed to check whether the list has length less than n?

9.5. Given the definition

$$\begin{array}{rcl} f([\,]) & = & [\,] \\ f(x :: xs) & = & (x * x) :: f(xs) \;\text{ when } x > 10 \\ f(x :: xs) & = & f(xs) \;\text{ when } x \leq 10 \end{array}$$

show an inside-out evaluation of expression f($[4, 12, 15, 6, 11]$). Show the entire expression at each step. Assume that arithmetic is done as soon as possible.

9.6. Show that the number of substitution steps needed to compute cat(x, y) is $2n + 1$, if n is the length of list x.

9.7. The cat function uses memory sharing. Show a linked list diagram of lists $[1, 2]$ and $[3, 4]$, as well as the result of computing cat($[1, 2]$, $[3, 4]$), showing the memory sharing. Is memory sharing safe for equational programming, or is it likely to cause trouble?

9.8. Suppose that the ascend function (Equations (9.8.3)) is defined as follows.

$$\begin{array}{rcl} \text{ascend}(x) & = & \text{true when length}(x) < 2 \\ \text{ascend}(a :: b :: r) & = & a < b \text{ and ascend}(b :: r) \end{array} \tag{9.11.1}$$

Assume that lists are represented in linked form, and that computing length(x) takes time proportional to the length of x. To within a constant factor, how long does it take to compute ascend(x), in the worst case, for a list x of length n?

9.9. For a positive integer k, say that next(k) $= k/2$ when k is even, and next(k) $= 3k + 1$ when k is odd.

(a) Write an equational definition of next.

(b) The *hailstone sequence* starting at n is a list where each number k is followed by next(k), until the list ends on 1. For example, the hailstone sequence starting at 13 is $[13, 40, 20, 10, 5, 16, 8, 4, 2, 1]$. Write an equational definition of hailstone(n), yielding the hailstone sequence starting at n.

(c) Define a function hailstoneLength(n) that returns the length of the hailstone sequence that starts at n. For example, hailstoneLength(13) = 10. Give a definition that uses the length and hailstone functions.

(d) Give a direct definition of hailstoneLength that does not use the length or hailstone functions.

9.10. Write an equational definition of function digitsum, where digitsum(n) is the sum of the digits in decimal number n. For example, digitsum(554) = 5 + 5 + 4 = 14. Use operators **div** and **mod** where x **div** y is the integer quotient when x is divided by y, and x **mod** y is the remainder.

9.11. Using your solution to Exercise 9.10, show an inside-out evaluation of digitsum(3942).

9.12. Write an equational definition of function seven, where seven(n) returns true if decimal number n has any 7's in it. For example, seven(372) = true and seven(29) = false.

9.13. Using your solution of Exercise 9.12, show an inside-out evaluation of expression seven(9725).

9.14. Write an equational definition of function dup where dup(n, x) produces a list of n copies of x. For example, dup(3, 20) = $[20, 20, 20]$.

9.15. Write an equational definition of a function stutter(x) that takes a list x and produces a list with each member of x duplicated. For example, stutter($[a, b, c, d]$) = $[a, a, b, b, c, c, d, d]$.

9.16. A *predicate* is a function that produces a boolean result. Write an equational definition of a predicate called prefix so that prefix(x, y) is true just when list x is a prefix of list y. For example, prefix($[1, 2]$, $[1, 2, 4, 6]$) is true, but prefix($[1, 2]$, $[1, 5, 2, 6]$) is false. The empty list is a prefix of every list, and every list is a prefix of itself.

9.17. Show an inside-out evaluation of prefix($[1, 2]$, $[1, 2, 3, 4]$) using your definition of prefix from the preceding question. Show the entire expression at each step.

9.18. The zip function takes a pair of lists and produces a list of pairs. For example, zip($[a, b, c]$, $[d, e, f]$) = $[(a, d), (b, e), (c, f)]$. (Compare that to the action of a zipper.) If the two lists have different lengths, zip goes only to the length of the shorter list, so zip($[a, b, c]$, $[d]$) = $[(a, d)]$. Write a definition of zip.

9.19. Write an equational definition of function listSum(x) that produces the sum of a list of numbers x. For example listSum($[1, 2, 3, 4]$) = 1 + 2 + 3 + 4 = 10.

9.20. This problem has you write another definition of function listSum from the preceding exercise.

(a) Using pseudo-code, write a loop that sums the members of a loop by adding the head of a list to a variable, and taking the tail of the list, each time around the loop.

(b) Show a table of the values of the loop control variables each time control reaches the top of the loop, for the example (listSum $[2, 4, 6]$).

(c) Find a *useful* invariant function of your loop. (If your loop control variables are s (a number) and y (a list), and the orignal list is called x, how are s, listSum(y) and listSum(x) related?)

(d) Write an equational definition of the invariant function.

(e) Write a definition of listSum in terms of the invariant function.

(f) Your definition of listSum is more involved that the one that you would have produced for Exercise 9.19. Does this new definition have any advantage over the shorter, more direct, one?

9.21. Chapter 10 defines function decimal, where decimal("247") = 247, using a tool called a fold. For this exercise, you will define decimal without a fold.

(a) Using pseudo-code, write a loop that computes decimal(s) for a string s of decimal digits. (A string is just a list of characters.)

(b) Show the values of the loop control variables each time the loop reaches its top when computing decimal("2468").

(c) Derive an invariant function from the table in the example. What useful thing is true at each line? (Hint: your invariant function might involve taking the digits of a given number and following them by the digits in a given string. For example, the digits of 24 followed by the digits in string "68" would be number 2468.)

(d) Convert the invariant function to a function definition.

(e) Define decimal in terms of your invariant function. If you cannot, then you might have made an inappropriate choice of invariant function.

9.22. Suppose that you already have a function called *get* that gets a decimal integer from a string. It produces an ordered pair holding the integer and the rest of the string, after the integer. For example, get("123 456 78") = (123," 456 78"), where string "123" has been converted into the integer 123. Before getting digits, get skips over leading blanks. Using get, write an equational definition of a function getAll that takes a string of decimal digits and blanks, and converts it into a list of all of the blank-separated numbers in the string. For example, getAll("123 456 78") = $[123, 456, 78]$. Assume that there are no spaces after the last digit.

9.23. A binary search tree is either an empty tree or is a node with a label and two subtrees, called the left subtree and the right subtree. If a node v has label x, then all labels in the left subtree of v are smaller than x, and all labels in the right subtree of v are greater than x.

Suppose that you have a binary search tree, and you want to produce a list of all of the labels in the tree in ascending (left-to-right) order. You have the following operations available: empty(t) is true if t is an empty tree; left(t), right(t) and label(t) are the left subtree, right subtree and label, respectively, of a nonempty tree t.

(a) Write an implementation of members(t), which produces a list of the members of tree t in ascending order. You can use concatenation (++).

(b) Write an implementation of members(t) that does not use concatenation. Instead, use a helper function membersHelp(t, r) that produces a list of all members of tree t, in ascending order, followed by list r. Define members in terms of membersHelp.

(c) What is the advantage of using membersHelp and avoiding concatenation?

9.24. Translate each of the following to Cinnameg and test it using a Cinnameg compiler.

(a) Equations (9.4.1) and (9.4.2).

(b) Equations (9.8.3).

(c) Equations (9.8.4).

9.25. The second equation in Equations (9.10.1) for the pow function is not really necessary, since the other equations cover all cases. Does the second equation help at all, or should it be removed? (**Hint.** How many multiplications does pow use to compute 3^{16}?)

9.26. Function sublist is defined so that sublist(x, y) takes a list x and an *increasing* list of positive integers y and produces the list of members of x in the positions indicated by y, where 1 means the first member, 2 the second member, etc. For example, sublist($[a, b, c, d, e, f, g]$, $[2, 3, 5]$) = $[b, c, e]$, since it asks for the second, third and fifth members of list $[a, b, c, d, e, f, g]$. If an index is requested that does not exist or that is out of order, then sublist should ignore that index.

(a) Write a definition of sublist in Cinnameg in an equational style, with type information and samples. You can use operator _, where x_n is the n-th member of list x.

(b) A more efficient approach for linked lists is to define a more general function sublistFrom(x, y, i) that is similar to sublist, but it assumes the first member of list x is numbered i, the next $i + 1$, etc. For example, sublistFrom($[b, c, d]$, $[3, 4]$, 3) = $[b, c]$. Write a definition of sublistFrom, then define sublist in terms of sublistFrom.

9.27. Say that list x is a sublist of list y if x can be obtained by removing zero or more members from list y. The empty list is a sublist of every list, and every list is a sublist of itself. A sublist preserves order. For example, $[1, 3]$ is a sublist of $[1, 2, 3, 4]$, but $[3, 1]$ is not. Write a Cinnameg definition of function isSublist(x, y) that returns true just when x is a sublist of y. Include type information and samples.

9.28. You do not always need to use recursion in a function definition. Without using recursion, write a Cinnameg definition of function rotate, which moves the first member of a list to the end. For example, rotate($[a, b, c, d]$) = $[b, c, d, a]$. Assume that rotate($[\,]$) = $[\,]$. You can use concatenation.

9.12. Bibliographic notes

Equational programming got its start with recursive function theory [32], before there were any modern computers. Languages that support an equational style include ML [78], Haskell

[15, 95] and Miranda [96]. Lisp [44, 92] is an early language that supports functional programming, but its unusual syntax tends to hide the equational nature of function definitions.

Chapter 10

Higher Order and Lazy Functions

10.1. Functions as values

This chapter covers some fairly abstract ideas. To keep it as concrete as possible, we do all examples in Cinnamegrather than relying on mathematical equations. It is important to realize that these ideas represent effective tools for writing software.

Until now, we have treated functions as tools to be defined and used. But you can do more with a tool than use it. You can put it in a tool box, lend it to a friend, or make improvements to it. In a program, that means treating a function as a first class value.

There is an important notational issue. You are probably accustomed to showing a parameter when discussing a function. For example, you might draw a graph of function $\sin(x)$. But in our notation, $\sin(x)$ is the result that the sin function produces on parameter x. The function itself is just called sin. Suppose that you have a function called longfunctionname, and you find it convenient to give this function a new, shorter name, s. So you write

```
{s(x) = longfunctionname(x)}
```

But if you realize that a function is just a value, you can write a slightly shorter definition.

```
{s = longfunctionname}
```

For this chapter, it is important to begin thinking of functions a values on the same footing as numbers and lists.

10.1.1. Function expressions

If a function is a value, then it should be possible to write an expression whose value is a function. In Cinnameg, if p is a pattern and e is an expression, then expression (p `|->` e) has a function as its value, namely the function f defined by $f(p) = e$. For example, a function that squares its input can be written (x `|->` $x*x$). Figure 10.1 shows two equivalent definitions of sqr, one using a function expression.

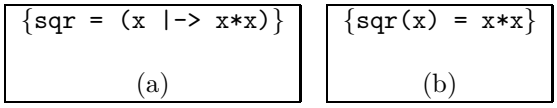


Figure 10.1: **Two equivalent definitions of the sqr function.** The Cinnameg compiler treats definition (b) as an abbreviation for definition (a). In general, if A, B and C are expressions, then {A B = C} is an abbreviation for {A = (B |-> C)

The process of running a function on a given parameter is called *applying* the function, and expression $f\,x$ produces the result of applying function f to parameter x. The function can be given by any expression that yields a function. For example, expression (x |-> $x*x$) 5 yields the result, 25, of applying function (x |-> $x*x$) to 5. You can add parentheses freely. Expressions $f\,x$, $f(x)$, $(f\,x)$ and $(f)(x)$ all have the same meaning.

10.1.2. The mechanics of function application

The *body* of a function expression is the part after |->. For example, the body of function (x |-> $x*x$) is expression $x*x$. Evaluation of expression ((p |-> b) v) is done in three steps: (1) solve pattern match equation $p = v$; (2) substitute the solution of that equation into the body b; and (3) evaluate the body with those substitutions. For example, expression ((x |-> $x*x$) 5) starts by solving the trivial pattern match equation $x = 5$ for x. Then it substitutes 5 for x into the body, yielding 5*5. Finally, evaluation of 5*5 yields the result, 25.

10.2. Curried functions

Suppose that function add is defined as follows.

```
{add = (x |-> (y |-> x+y))}
```

Clearly, add is a function. What do you get if you evaluate add(2)? To find out, follow the rules for applying a function. The body of (x |-> (y |-> $x+y$)) is (y |-> $x+y$), and substituting $x = 2$ into the body yields expression (y |-> $2+y$). So add(2) is a function, namely the function that returns a number that is 2 larger than its parameter. For example, (add(2))(3) = 5 and (add(2))(7) = 9. To define a function called shove so that shove(x) = $x+2$, it suffices to write

```
{shove = add(2)}
```

Since {A B = C} is an abbreviation for {A = (B |-> C)}, the definition of add can be written as follows.

```
{add x = (y |-> x + y)}
```

Now, using the same idea again, you get

```
{add x y = x + y}
```

```
{one
 case take 0 xs          = []
 case take n []          = []
 case take (n+1) (x::xs) = x :: take n xs
}
```

Figure 10.2: **A definition of take as a curried function.** Parentheses around $(n+1)$ and $(x :: xs)$ are important, since juxtaposition has higher precedence than binary operators.

Juxtaposition (putting things next to one another) associates to the left, so expression add x y is implicitly understood to be the same as ((add x) y).

Function add appears to have two parameters. But *all* functions in Cinnameg really have just one parameter. The add function takes one parameter and returns a function. Initially, add is waiting for its first parameter, x. When you evaluate add(2), you get a function that is remembering that x is 2, and is waiting to get the value of y. Functions defined in this way are called *curried* functions.[1]

There is an analogy between curried functions and multi-dimensional arrays. Some languages, including C++ and Java, treat a two-dimensional array as an array of arrays. If you declare

```
int A[5][5];
```

in C++ to create a 5×5 array, then you can use two indices, such as $A[2][3]$, to get a particular integer from the two-dimensional array, or you can use just one index, such as $A[2]$, to get the entire row at index 2. A is a two-dimensional array of integers, $A[2]$ is a one-dimensional array of integers and $A[2][3]$ is an integer. Analogously, add is a function that takes two numbers and produces a number, (add 2) is a function that takes one number and produces a number, and (add 2 3) is a number.

You can define any function that you think of as taking more than one parameter in a curried form. For example, function take from Chapter 9 gets a length n prefix of a given list x. It naturally has two parameters, n and x. Definition of take as a curried function is shown in Figure 10.2. Now (take 3) is a function that extracts a length 2 prefix of a given list, and (take 2) $[2, 4, 6, 8] = [2, 4]$.

10.3. Some tool-building tools

A higher-order function is one that either takes another function as a parameter or produces another function as its result, or both. A higher order function typically does not implement just a single algorithm, but an entire class of algorithms. Think of it as a tool that builds tools for you, making it possible to write the details of the entire class of algorithms just once and reuse it many times.

[1] Curried functions are named for Haskell Curry, one of the early researchers in the foundations of functional programming. Curry used this style, but claimed that it was invented much earlier

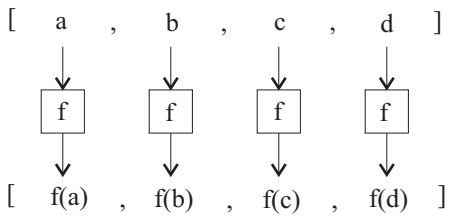


Figure 10.3: **Diagram of a map.** The diagram shows computation of map f $[a, b, c, d]$.

10.3.1. Mapping a function onto a list

Higher order function *map* is defined so that (map f [a,b,c]) = [$f(a)$, $f(b)$, $f(c)$]. In general, map applies f to each member of a list separately. A simple definition, using a list comprehension, is

```
{map f x = [f(y) | y from x]}
```

For example, map sqr $[2, 3, 4, 5] = [4, 9, 16, 25]$ and map (+) [(2,3), (4,5)] = $[5, 9]$ and map (take 2) $[[a, b, c, d], [e, f, g]] = [[a, b], [e, f]]$. Figure 10.3 shows how map works in a graphical form. Map captures all algorithms that work by modifying each member of a list independently of the other members. Examples of functions that are maps are (1) converting all letters in a string to upper case, (2) replacing each file in a list by the length of the file, and (3) decoding each string in a list of strings.

10.3.2. Checking whether any member of a list satisfies a predicate

Another useful higher order function is *someSatisfy*, where (someSatisfy f $[a, b, c]$) is true if at least one of $f(a)$, $f(b)$ or $f(c)$ is true. For example, (someSatisfy (x `|->` x $>$ 100) $[20, 300, 5]$) is true, but (someSatisfy (x `|->` x $>$ 100) $[1, 2, 3, 4]$) is false. Function someSatisfy captures all algorithms that search a list looking for a particular kind of value. For example, asking whether a value is a member of a list is a search problem. The following defines a curried member predicate, where (member k x) is true if k is a member of list x, using someSatisfy.

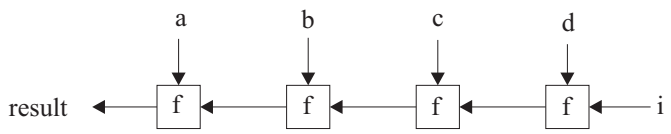
```
{member k = someSatisfy (y |-> y == k)}
```

You might find a slightly longer definition easier to understand.

```
{member k x = someSatisfy (y |-> y == k) x}
```

Definition of someSatisfy is straightforward.

```
{case someSatisfy f []      = false
 case someSatisfy f (x::xs) = f(x) or someSatisfy f xs
}
```




```
{one
 case foldRtoL i ? []      = i
 case foldRtoL i f (x::xs) = f(x, foldRtoL i f xs)
}
```

Figure 10.4: **Definition of foldRtoL and a diagram showing computation of foldRtoL** i f [a,b,c,d]**.** Pattern ? matches anything, and is a don't-care value. The computations done in the diagram are $f(d,i)$, $f(c,f(d,i))$, $f(b,f(c,f(d,i)))$ and the final result is $f(a,f(b,f(c,f(d,i))))$.

10.3.3. Folding, or scanning, a list

Some algorithms on lists work by scanning once over the list, either from left to right or from right to left, and producing an answer at the end of the scan. While scanning the list, the algorithm remembers some information, such as the largest value seen, or the sum of the values seen. Scan algorithms are called *folds* of lists.

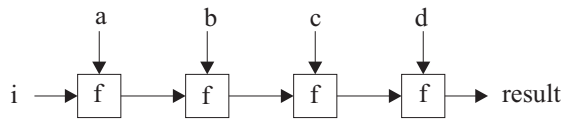
Two higher order functions will capture the two classes of scan algorithms; foldRtoL scans from right to left and foldLtoR scans from left to right. Each function needs three parameters: an initial value, representing the initial information that the algorithm remembers; a function that updates the information as new list members are encountered; and a list to scan. There are arbitrary decisions to be made in defining foldRtoL and foldLtoR: in what order, and in what way, should the parameters be given to these functions? We will use a definition of foldRtoL where foldRtoL i f [a,b,c] $= f(a,f(b,f(c,i)))$. The initial value is i and the update function is f. It is easier to see what foldRtoL does when the function f is replaced by a binary operator, such as $+$. Then foldRtoL i (+) [a,b,c] $= a+b+c+i$, and foldRtoL i (*) [a,b,c,d] $= a*b*c*d*i$, where in both cases the operations are performed from right to left. So, for example, foldRtoL 0 (+) [2,3,4,5] $= 2+3+4+5+0 = 14$. A function to sum the members of a list is as follows.

```
{sum = foldRtoL 0 (+)}
```

Figure 10.4 shows a definition of foldRtoL and a diagram of its action. Function foldLtoR is similar to foldRtoL, but it scans the list from left to right. Figure 10.5 shows a diagram of the action of foldLtoR i f $[a,b,c,d]$, along with the definition.

Understanding folds

There is a general rule for defining functions using folds. As you scan a list, you remember some information about the values that you have seen so far. At each stage of the fold, the value to remember is the answer that would be produced if the part of the list looked as so far were the entire list. For example, you know that sum[3,4,5] = 12, so the sum algorithm




```
{one
 case foldLtoR i ? []      = i
 case foldLtoR i f (x::xs) = foldLtoR (f(i,x)) f xs
}
```

Figure 10.5: **Computation and definition of a left-to-right fold.** The diagram shows computation of foldLtoR i f [a,b,c,d] by computing $f(i,a)$ then $f(f(i,a),b)$ then $f(f(f(i,a),b),c)$ and finally the result $f(f(f(f(i,a),b),c),d)$. Notice that the definition of foldLtoR is tail-recursive.

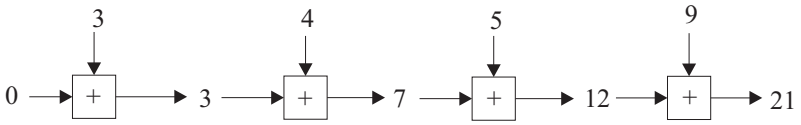


Figure 10.6: **Summing the numbers in a list.** Notice that the partial result, as the algorithm scans the list, is the sum of the numbers seen so far. It is a typical characteristic of scan algorithms that the partial results are just what the final results would be if the scan had reached the end of the list.

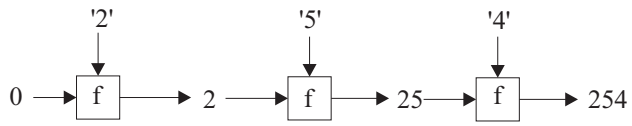


Figure 10.7: **Converting a string of decimal digits to an integer using a left-to-right fold.**

```
{decimal = foldLtoR 0 f |
  {f(x,c) = 10*x + digitValue(c)}
}
Example decimal "2531" = 2531.
```

Figure 10.8: **Definition of decimal with embedded function definitions.** Expression $e|d$ produces the value of e after performing definitions d.

will remember 12 after reading prefix [3,4,5] of a longer list. Figure 10.6 shows the action of sum as a left-to-right fold.

As another example, suppose that you would like to convert a string of decimal digits to a number. Say, decimal("254") = 254. A scan, from left to right, should do the job. After reading no digits, the result should be 0. After reading "2", the result should be 2, and after "25", number 25 should be produced, as shown in Figure 10.7. The only question is how to write function f. Examining the diagram, you can see that $f(0, \text{`2'}) = 2$, $f(2, \text{`5'}) = 25$ and $f(25, \text{`4'}) = 254$. The algorithm needs to convert a character such as '2' to its associated number, 2. Cinnameg function digitValue will do that, and $f(x, c) = 10 * x + \text{digitValue}(c)$ will work in the fold. Function decimal can be defined as follows.

```
{decimal = foldLtoR 0 ((x,c) |-> 10*x + digitValue(c))}
```

Often, definitions that make use of higher order functions need helper functions, as does the definition of decimal, and it can be convenient to give those helper functions names. Figure 10.8 shows a definition of decimal with f defined inside it.

10.4. Write-only programs?

Powerful operations such a map and folds take quite a bit of thought to use. If you use them extensively, and especially if you use several of them in one line, you can get a definition that is nearly impossible to read. For example,

```
{isPrime n =
  n > 1 and notf someSatisfy (k |-> n `mod` k == 0) [2,...,n-1]
}
```

defines a function isPrime that tells whether an integer is prime. But that is certainly not obvious. Because of this tendency toward unreadability, programs written in this style have

been called write-only programs. To avoid the problem, use only one, or occasionally two, powerful operations per line. Carefully document what you have done at each line. Show, with examples, what is happening. Choose descriptive names. Then definitions can be made simple and readable.

```
%% (factorOf n) k is true if k is a factor of n.
{factorOf n k = (n `mod` k == 0)}
Example factorOf 10 5 = true.

%% isPrime(n) is true if n is prime.
{isPrime n =
  n > 1 and not(someSatisfy (factorOf n) [2,...,n-1])
}
Example isPrime(13) = true.
Example isPrime(9) = false.
```

Computer programs should be written with careful thought to each line, not by typing as fast as you can. Higher order functions tend to concentrate more thought into each line.

10.5. Lazy evaluation

We have seen in Chapter 9 that expressions can be evaluated in different orders, and that an evaluation policy chooses from among the possible orders. Inside-out evaluation, also called *eager* or *strict* evaluation, performs evaluations of all arguments of a function before evaluating the function call. Inside-out evaluation also goes by the name *call-by-value*, since the argument in a substitution is always a precomputed value.

Eager evaluation is a natural way to do things, but it is not always the best idea. For example, the conditional function, cond(a, b, c), naturally wants to have a evaluated first, followed by evaluation of either b or c, but not both. Boolean operations have similar improvements. For example, (x and y) is equivalent to cond$(x, y, false)$, suggesting that y should only be evaluated when x is true. These special functions seem to prefer outside-in evaluation (also called *call-by-name*), where arguments are kept unevaluated until they are needed. But to employ outside-in evaluation *always* also has problems. If function sqr is defined by sqr$(x) = x{*}x$, then computation of sqr$(g(4))$ by outside-in evaluation leads to expression $g(4) * g(4)$, and you find yourself evaluating the same expression $g(4)$ twice.

You might ask whether the advantages of inside-out and outside-in evaluation can be combined into a single policy that has the best characteristics of both. It turns out that there is a policy that *almost* does that. It goes by several names, including *lazy evaluation*, *nonstrict evaluation*, and *call-by-need*. The idea is that an expression should not be evaluated until its value is needed. An unevaluated expression is stored as a function, called a *promise*, that can compute the value of the expression when and if it is needed. For example, expression head(x) is stored as a parameterless function p where $p(\,) =$ head(x). The value x is built into function p.

To use this idea, the expression evaluator must look at each value, when it is needed, to see whether it is a promise. If so, then the evaluator *forces* the promise; it runs the function stored in the promise to find the expression value. The evaluator then performs a step, called *memoizing*, that is critical to the efficiency of lazy evaluation; it replaces the

promise by the value that was computed. If the same value is used again, no evaluation of a promise will be needed — the value will be there to get.

A promise is never copied. Instead, if it needs to be duplicated, two *references*, or *pointers* to the promise are created. That way, if any of the references need to evaluate the promise, all of them will see the answer after it is memoized.

10.5.1. Lazy evaluation in programming languages

With respect to evaluation mode, there are two classes of languages. *Nonstrict* languages, such as Haskell, use lazy evaluation as the normal policy. As mentioned above, lazy evaluation *almost* combines the best of inside-out and outside-in evaluation, so it is a sensible policy for purely functional languages, where nothing ever changes. The only issue is the need to build promises, which takes time and memory. Compilers for nonstrict languages can recognize cases where values will definitely be needed right away and use eager evaluation for those cases.

Strict languages, which include most programming languages, instead use eager evaluation as the normal policy, with some special cases where outside-in evaluation is employed. Imperative languages are almost always strict, because, when expressions can have side effects, delaying evaluation can cause the value produced by an expression to depend on when the expression is forced. Some strict languages provide a way for you to request delayed evaluation. Cinnameg, for example, is strict, but you can request delayed evaluation by enclosing an expression in *smiles*, as (:*e*:). Evaluating expression (:*e*:) yields a promise. Definition

```
{hd(x) = (:head x:)}
```

makes hd(x) the same as head(x), but it waits until the head is needed to perform the evaluation. (That might avoid an error, in the case where x is an empty list, if the value is never used in that case.) You can also wrap smiles around the cases in a definition by cases. For example,

```
{(:case take 0      ?        = []
   case take ?      []       = []
   case take (n+1) (x :: xs) = x :: take n xs
 :)
}
```

defines a function similar to the take function defined in Figure 10.2, but evaluation of (take n x) produces a promise. If you compute head(take 3 [2,4,6,8,10]), that promise will be forced, producing a list whose head is 2 and whose tail (computed by (take 2 [4,6,8,10])) is another promise. Only as much of the result list that is examined is computed.

Lazy evaluation is a powerful tool, and the next few sections illustrate some of its uses. The examples are written in Cinnameg, using smiles to request lazy evaluation.

10.5.2. Infinite lists

Lazy evaluation lets you produce infinitely long lists. Suppose that you would like to create the infinite list [1,2,3,4, . . .] holding all of the positive integers. You cannot store the entire list in explicit form. But you can store it implicitly, using lazy evaluation, as long as you never try to look at the entire list. We define positives = ints(1) where ints(n) = [n, $n+1$, $n+2$, . . .].

```
{ints(n) = (: n :: ints(n+1) :)}
{positives = ints(1)}
```

At first, list positives is represented by a promise p_1 that will compute it, if it is ever asked for. Suppose you ask for the head of positives. Forcing promise p_1 yields list 1::p_2 where p_2 is a promise that will compute list [2,3,...], if it is ever asked for. Asking for the second member of list positives replaces p_2 by a little bit more, so that list positives has the form 1::2::p_3, containing yet another promise p_3. List *positives* appears to be changing each time an evaluation of one of its promises is performed. But that is not really what is happening. Chapter 4 shows that there can be many different representations for the same value. A promise is a kind of representation of a value, and all that happens when a promise is forced is that the representation is changed. The value remains the same. The value of positives is constant, the infinite list [1,2,3,...].

Cinnameg provides a notation [n, ...] for the infinite list [n, $n+1$, ...], so you can define positives = [1,...]. List comprehensions in Cinnameg (Section 9.10.4) are lazy, and they can also be used to produce infinitely long lists. List comprehension

```
[2*n | n from [2, ...], prime?(n)]
```

yields the infinite list [4,6,10,14,...] of the doubles of all prime numbers.

10.5.3. Explicit lists for implicit searches

Suppose that you want to find the smallest prime number that is greater than or equal to n. In C, you write a loop. (Operator ! is the negation (*not*) operator in C.)

```
k = n;
while(!prime(k)) k = k + 1;
```

What that loop is doing is searching through an *implicitly* defined sequence, given by an initial value (n), and a way to obtain the value that follows any given value (just add 1). The program walks through the sequence without ever writing the whole sequence down at once. An alternative is to search an explicit list. For example, Cinnameg function nextprime defined by

```
{nextprime(n) = select prime? [n,...]}
```

does the job. (Expression (select p xs) produces the first member x of list xs such that $p(x)$ is true.) Only as much of the list as is needed is produced.

Object-oriented languages that provide collections such as sets and tables typically also provide objects called *iterators* that allow you to look at each thing in the collection. A simple iterator has a function init( ), which produces the initial value, a function next(x), which produces the value that follows x in the sequence, and a predicate last(x), which tells whether x is the last value in the sequence. A search for a value in a collection that has property p might look as follows, in general.

```
k = init();
while(!last(k) && !p(k)) k = next(k);
```

From a computational standpoint, the three functions init, next and last are adequate to define the search sequence implicitly. But, from a logical standpoint, you might ask whether this is the most pleasant way to provide a sequence to the programmer. Typically, for every

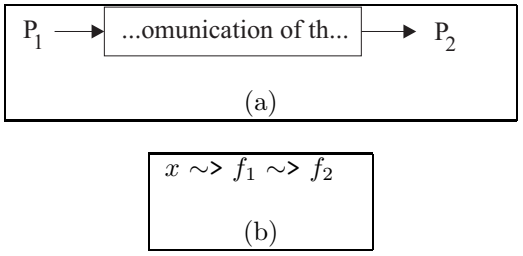


Figure 10.9: **(a) Two processes P_1 and P_2 communicating over a pipe, and (b) two functions f_1 and f_2 communicating by sending the result of f_1 as the argument of f_2.** The pipe only holds information that has been written by P_1 but not yet read by P_2. If f_1 produces a list, and if it is designed to be lazy, then communication between f_1 and f_2 behaves similarly to the pipe, and requires little memory, even if the list is long. The ~> notation is from Cinnameg: x ~> f is the same as $f(x)$, and is done from left to right.

kind of collection object, there is also a kind of iterator, increasing the size of the library. Also, everything that you want to do with an implicit sequence needs to be defined, typically with a loop; you cannot just use existing tools that work on explicit sequences.

An alternative is to produce an explicit list of all members of the collection, making all list functions in the library available for working with the sequence, without the need to rewrite them for iterators. Lazy evaluation is critical for doing that, since you do not want to produce the entire list if only a small part of it is going to be examined. You can view the presence of iterators in a language as compensation for the language's inability to handle lazy lists.

10.5.4. Memory and pipes

Operating systems provide a feature called a *pipe* for communicating between processes or threads. A pipe is a channel into which one process writes, and from which another reads. If the pipe is empty, then a process that tries to read from the pipe is suspended until another process writes something into the pipe. If the pipe gets too much information in it, then any process that tries to write into the pipe is suspended until another process consumes some characters from the pipe. That way, it is possible for two processes to exchange a great deal of information using a relatively small memory buffer to hold what is currently in the pipe.

Lazy lists provide a similar mechanism within a program. Suppose function P (the producer) creates a lazy list — one where only as much as is needed is computed. Suppose function C (the consumer) consumes that list, possibly searching for a value, or accumulating some value as it scans the list. As function C runs, more and more of the list is computed by P, and, because the results of lazy evaluation are memoized, what has been computed occupies memory. But eventually the garbage collector runs, the memory occupied by that part of the list that C has already passed over can be reclaimed, and that memory can be used again to produce more of the list. You can scan a long list with a small amount of memory.

10.5.5. The robustness of substitution

A fundamental principle of mathematics is that two equal expressions are interchangeable. If you have established that $x = y + z$, then you are allowed to substitute $y + z$ for x, or the other way around, in any other expression, as long as you are careful to parenthesize to avoid problems with operator precedence and associativity. Functional programming is designed to follow that principle as well. The only issue should be one of efficiency. That principle suggests that expression `(0 | {x = f(1)})` must produce result 0, the result of replacing all occurrences of x by $f(1)$ in expression 0. But in Cinnameg, definitions are eager, and expression `(0| {x = f(1)})` evaluates $f(1)$ before moving on; if evaluation of $f(1)$ runs forever, the expression will never produce result 0. So, with eager evaluation, performing a substitution can change the result of an expression from an infinite computation to an actual result and the other way around.

Lazy evaluation comes to the rescue. Cinnameg offers another kind of definition, {x `=*` e}, which is lazy; it creates a promise for e, binding x to that promise. Expression `(0 | {x =* f(1)})` does have value 0, since $f(1)$ will not be computed at all.

You can see the same thing at work when a conditional expression is used. From a mathematical standpoint, you should be able to give names to common subexpressions that must have the same value. The substitution rule suggests that expression `If* r == 0 then f(z) else If* r > 0 then 2*f(z) else 0 %If* %If*` should always produce the same result as `{ y =* f(z)} If* r == 0 then y else If* r > 0 then 2*y else 0 %If* %If*` And it does, but if the lazy definition is replaced by an eager one, then the two are not always the same, since an infinite computation of $f(z)$ will be trouble when $r < 0$.

When all computations are as lazy as possible and expressions with side effects are not allowed, the substitution rule is always correct, and the only effect that substitutions can have are on efficiency, program size and readability.

10.6. Exercises

10.1. Determine the value of each of the following Cinnameg expressions.

(a) map (x `|->` x + 3) [3, 7, 5]

(b) map (add 1) [2, 4, 6, 8]

(c) map (map (x `|->` x + 1)) [[2,3,4], [ ], [5, 6]]

(d) map tail [[2, 5], [1, 3, 5, 7]]

(e) someSatisfy (x `|->` x) [false, true, false]

(f) map (someSatisfy (y `|->` y > 10)) [[2, 3, 4], [ ], [10, 20, 30]]

(g) foldRtoL [ ] (::) [6, 3, 1]

(h) foldRtoL [ ] (++) [“choc”, “o”, “late”]

(i) foldLtoR 1 (+) [2, 3, 4]

10.2. Given definition {`f x y = (x+1)::y`}

(a) What kind of thing is x (a number, a list or a function)?

(b) What kind of thing is y (a number, a list or a function)?

(c) What kind of thing is $f\,x\,y$ (a number, a list or a function)?

(d) What kind of thing is $f\,x$ (a number, a list or a function)?

10.3. Given definition {`f x y z = y x (x::z)`}

(a) What kind of thing is parameter z? (A number, a list, a function, or something else)?

(b) What kind of thing is parameter y? (A number, a list, a function, or something else)?

(c) What kind of thing does expression $f(2)$ yield? (A number, a list, a function, or something else? Is this expression allowed?)

10.4. Cinnameg library function *select* is written so that (select p x) returns the first member m of list x such that $p(m)$ is true. Notice that p is a predicate. Predicate odd? returns true on an odd number and false on an even number. What is the value of Cinnameg expression map(select odd?)[[2, 3, 4, 5], [1, 2, 3, 4], [2, 4, 6, 7, 8]]?

10.5. Function select of the preceding exercise has the difficulty that it might not have anything to select. For example, it is not clear what (select (x `|->` x $<$ 0) [2,3]) should yield. Suppose that (selectD d p L) yields the same result as (select p L) as long as some member of list L makes predicate p produce true, and yields d otherwise. Write a definition of selectD.

10.6. Show how to write a definition of function doubleAll(L), which takes a list L of numbers and produces a list of the doubles of those numbers. For example, doubleAll($[5, 2, 19, 3]$) = $[10, 4, 38, 6]$. For this definition, do not use any direct recursion. Use map to do the work.

10.7. Show how to write the map function without using a list comprehension, but using recursion.

10.8. Cinnameg library function filter selects all members of a list on which a given predicate yields true. For example, filter (x `|->` x $>$ 20) $[30, 2, 7, 85, 6, 23]$ = [30, 85, 23].

(a) Write a definition of filter using a list comprehension.

(b) Write a definition of filter without using a list comprehension, using recursion instead.

10.9. Show how to implement each of the following as a fold, either from left to right or from right to left (your choice). Do not use recursion.

(a) The factorial function.

(b) The list concatenation function, cat. (Does it matter whether you do a right-to-left fold or a left-to-right fold?)

(c) The map function.

(d) The someSatisfy function.

(e) The list reversal function.

(f) The sort function of Figure 9.10. (Just define sort as a fold, not insert.)

(g) The function maxlist m L that produces the largest member of list $m :: L$.

10.10. The inner product ip(x, y) of two lists of numbers x and y is defined so that ip($[a, b, c]$, $[d, e, f]$) $= a * d + b * e + c * f$.

(a) Write a definition of function sum, defined in Exercise 9.19, using a fold function.

(b) Write a definition of function zip, defined in Exercise 9.18, using recursion.

(c) Write a definition of function products, where products($[a, b, c]$, $[d, e, f]$) $= [a * d, b * e, c * f]$. In general, products takes a pair of lists and makes a list of the componentwise products. Use zip and map to do this.

(d) Write a definition of ip using sum and products. It should be clear from the definition that ip is a sum of products.

10.11. In a purely functional language, is it ever possible to compute the same expression twice in the same context and get different values? For example, if E is an expression, and you compute E two times, one right after another, could the first computation yield a different result from the second computation? Why or why not?

10.12. Answer the preceding question for an imperative language such as Java or C++. Explain your answer.

10.13. The composition $f \circ g$ of two functions f and g is defined by $(f \circ g)(x) = f(g(x))$. Suppose that you would like to compute the composition of a list of functions. For example, composeAll $[f, g, h] = f \circ g \circ h$. Show how to do this without using recursion using a fold. Does it matter whether the fold is from left to right or from right to left? What function should you use as the initial value of the fold? That function determines what composeAll [ ] is.

10.14. Sorting can be done with respect to any comparison operation. If $\ll$ is a given comparison relation, the goal is to rearrange a list into an order $[a_1, \ldots, a_n]$ where $a_1 \ll a_2 \ll \cdots \ll a_n$. For example, if you sort using comparison operation $\leq$, then you sort into ascending (or nondescending) order. If you sort with respect to comparison $\geq$, you sort into descending (or nonascending) order. Modify the definition of quicksort shown in Figure 9.9 so that, in addition to the list, it takes a comparison predicate as a parameter, and sorts with respect to that comparison predicate. The comparison predicate takes an ordered pair and returns a boolean result. (Assume that the comparison predicate returns true on equal values.) Make your function curried, with the comparison function first, so that it is easy to define functions that sort in specific orders.

10.15. Write a Cinnameg definition of function infinitedup(x) that produces the infinite list $[x, x, x, \ldots]$.

10.16. Write a Cinnameg definition of function primes(n) that produces a list of all prime numbers that are greater than n.

10.17. Consider the following definition written in Cinnameg.

```
{f(n) = (:n*n :: f(n+1):)}
```

What value does expression f(2) compute? Give the full value, as it would be if it were examined. (It suffices to give a clear description of the full value.)

10.18. Two useful concepts that you have learned about mathematics are *substitution* and *specialization*. If you know that $z = y + u$, the substitution rule says that you can replace z by $y + u$, and that you can also replace $y + u$ by z. If you know that $x + 1 = g(x)$ for every x, then the specialization rule says that you can replace x by anything in that equation and get another true equation. For example, you know that $5 + 1 = g(5)$.

Suppose that you are given the following equations, the first true for all values of x and y.

$$\begin{aligned} \text{add } x\ y &= x + y \\ \text{inc} &= \text{add } 1 \end{aligned}$$

Using only substitution and specialization, show that inc 3 = 1 + 3. Be careful to remember that function application (juxtaposition) associates to the left, and that it is not associative. That is, $(a\,b)\,c \neq a\,(b\,c)$. Explain, at each step, which rule you are using.

10.19. A directed graph has vertices numbered 1, ..., n. The graph is represented by a list $[L_1, \ldots, L_n]$ where L_i is a list of all vertices v such that there is a directed edge from vertex i to vertex v. For example, list [[2], [1,3], [2,3]] describes a directed graph with vertices 1, 2 and 3 and with directed edges (1,2), (2,1), (2,3), (3,2) and (3,3).

Write a definition of function accessible(G, v) that takes a graph G and a vertex number v, and produces a list of all vertices u such that there is directed path from v to u.

10.20. A group of people needs to walk across a rickety bridge. The people walk at different speeds, and you are given a list of the number of minutes each person takes to cross the bridge. For example, list [2,3,6,10] indicates that one person can cross in two minutes, the next in three minutes, etc.

Because the bridge is rickety, only two people can be on the bridge at once. To make matters worse, it is night, and anyone crossing the bridge needs to carry a flashlight. But there is only one flashlight. So two people will cross the bridge with the flashlight, then one will carry the flashlight back, then two people will cross, and so on, until everyone is across. When two people cross together, they go at the speed of the slower person.

You would like to determine the least number of minutes that it takes for all of the people to cross the bridge. If the list is [1,100,100,100], it takes 302 minutes for everyone to cross the bridge. If the list of times is [1,2,5,10], then all can get across in 17 minutes. Do not proceed until you see how each of those times is achieved. This problem is much more subtle that it looks initially, and a naive algorithm will not work.

(a) Write a function (minOver f L) in Cinnameg so that minOver f [a,b,c,d] = min[$f(a)$, $f(b)$, $f(c)$, $f(d)$]. But make minOver work for any nonempty list,

not just for lists of length four. Cinnameg provides a standard function, min, that produces the smallest member of a given nonempty list. For example, min[2,4,1,5] = 1.

(b) Suppose that the people start on the left side and need to cross to the right side. Write a function that takes a pair (L_1, L_2), where L_1 is a list of the crossing times of the people on the left side, and L_2 is a list of the crossing times of the people on the right side, and returns the total amount of time necessary for everyone represented in list L_1 to cross the bridge. First, work out the times for lists L_1 of length no more than two. Then, for longer lists, try all possible pairs of people to walk across the bridge together, and select the pair that leads to the least total time. Two calls to minOver (one to select the first person, the other to select the second person), plus a recursive call, will do the job. (To select a person to walk back, it can be shown that you can always select the fastest walker who is on the right side of the bridge. But be careful to select the fastest one after two people have crossed from left to right.)

Arrange only to pass your function lists that are sorted into nondescending order. You will find standard operator $-/$ useful ($L - /y$ is list L with the first occurrence of y removed), and you will also find the insert function from Figure 9.10 useful. Test your function.

(c) You will find that your function from part (b) is *very* slow on lists of size about 7 or 8. Try it to see. The problem is that it computes the same result over and over. Read about the *memoize* function in the Cinnameg documentation, and use it to improve your function. (This is why you should keep the lists of times sorted. You do not want to recompute the cost of list [2,7,4,1] when you already have a result for [1,2,4,7].) Test your modified function. Does it run faster?

10.21. Suppose that an iterator for a collection is defined by functions init( ), next(x) and last(x). Show how to examine the sequence backwards, finding the last value y that makes $p(y)$ true. How would you do the same job if given an explicit list?

10.22. Write a lazy list concatenation function. It should be able to produce the concatenation of two lists, but should not spend more time than is necessary computing it. If the list A is infinitely long, then the concatenation of A and B should be the same as A; your concatenation function must not get stuck evaluating A. (The Cinnameg operator ++ is lazy. For this exercise, do not use it.)

10.23. Write a lazy filter function, where filter p L produces a list of all members x of list L such that $p(x)$ is true. (So p is a predicate.) It should be possible to filter an infinitely long list.

10.24. Suppose that L is a list, and that function f takes a list and produces another list. You intend to compute $f(L)$, and you know you will need to use the entire result list. Is there any point in function f being lazy, or would it always make more sense for f to be eager? Think of how the result of $f(L)$ might be used, and the memory requirements for the entire computation.

10.25. Give an example where the substitution rule fails dramatically in an imperative language by showing an expression E that contains two identical subexpressions, but

if you derive E' from E by giving a name to the result of the common subexpression and using the name twice, then E' produces a different result from E. (Both E and E' should produce a result, not loop forever.)

10.26. Some compilers, both for functional languages and imperative languages, perform an improvement called common subexpression elimination, where a subexpression that occurs more than once is only computed once for efficiency. Improvements must be sure not to change the meaning of a program. What tests does a common subexpression eliminator for an imperative language need to make? What about for a functional language?

10.27. A list comprehension has the form $[E \mid A_1, A_2, \ldots, A_n]$ where each of $A_1, \ldots, A_n$ is either a *source component* of the form p *from* L, for some pattern p and expression L, or is a *filter component* that is an expression of type Boolean. A_1 is required to be a source component. Give a general way of converting a given list comprehension into an equivalent expression that does not involve list comprehensions. (If you write `Advise echo.` just before a definition in a Cinnameg program, the compiler will show the internal form that it has generated. If you encounter difficulties you might find that useful.)

10.7. Bibliographic notes

Darlington Henderson and Turner [34] describe use of higher order functions, as do the references from the preceding chapter. The analogy between curried functions and arrays is from Paulson [78].

Haskell [15, 95] and Miranda [96] are examples of languages that use lazy evaluation throughout. Algol 60 [84], an imperative language, uses a calling mechanism referred to as call-by-name for passing parameters to functions. With call-by-name, each time you use a parameter, you re-evaluate the expression that was passed as that parameter, so it is similar to outside-in evaluation, without memoization.

Chapter 11

Scheme

11.1. The Lisp family

The Lisp (LISt Processing) family of languages was begun by John McCarthy, with the first implementation of Lisp available in 1960. All of the languages in the Lisp family have a few common characteristics.

1. Lists are the basic data structure. The operations that we have called head, tail and :: are called *car*, *cdr* and *cons*, respectively, in Lisp. (Cdr is pronounced could-er.) For the remainder of this chapter we use terms car and cdr instead of head and tail.

2. The major paradigm of the language is functional, and recursion is the main control mechanism, but other paradigms are supported to some extent. It is possible, for example, to write a loop, and to change a variable.

3. The syntax is extremely simple. Lists are written in parentheses. The members of a list are written after one another, without any additional syntax such as commas.

4. Data and programs are of the same form. A program is a list. As a result, it is easy to treat data as a program, or to treat a program as data.

5. There is no concept of a type when a program is compiled. All type checking is done at run time.

During the 1960s and 1970s, Lisp became popular as a language for symbolic computing. Several different dialects were developed. In an attempt to consolidate the common features of these dialects, and to standardize the language to some extent, Common Lisp was defined. Today, Common Lisp is a popular language for symbolic computing.

The idea of Common Lisp was not to restrict dialects, but to suggest a core language that all dialects of Lisp should support, and possibly extend. But, even though it is intended to be a common core, Common Lisp is still a large language. An alternative Lisp dialect, Scheme, does not contain the Common Lisp core, and instead is designed to be small and elegant. The philosophy of Scheme is expressed well in the first paragraph of the Sheme report: "Programming languages should be designed not by piling feature on top of feature, but by removing the weaknesses and restrictions that make additional features appear necessary."

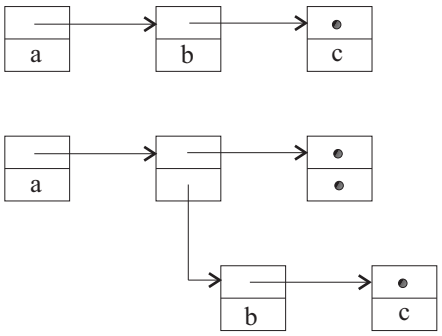


Figure 11.1: **Diagrams of the representation of lists (a b c) and (a (b c) ( )).**

11.2. The syntax of Scheme and the meaning of lists

An identifier, or name, can be any sequence of letters, digits and certain special characters, provided it does not begin with a digit or a period or the special symbols +, − or #. (The single-character names + and − are exceptions; they are identifiers.) The special symbols that can occur in identifiers are (`! $ % & * + - . / : < = > ? @ ^ _ ~`). For example, `this-identifier!` and `*+/` can both be used to name something. There are a few reserved words, such as **if**, **let** and **define**, that cannot be used for names.

Write numbers, such as 34 and 16.23 in usual form. Boolean values are #t (true) and #f (false). The character a is written #\a, the space character is written #\space, and the newline character is #\newline. Strings are enclosed in double quotes, such as "abc". In Scheme, a string is not considered to be a list.

One of the most useful kinds of data in Scheme is a *symbol*, which is just a name, without quotes. Symbols have an advantage over strings that equality testing is very efficient for them. A symbol is stored as an integer symbol-identity. To ask whether two symbols are the same, you just compare their identities, rather than comparing strings character-by-character. To get a symbol in a Scheme program, put a single quote mark in front of it. For example, 'camel is the symbol **camel**.

The syntax of Scheme is very simple. To write a list in Scheme, enclose the members of the list in parentheses, and separate them by spaces. For example, (a b c) is a list of three things. The empty list is written ( ). A list can contain lists; (a (b c) ( )) is a list of three things, the second and third of which are lists. Scheme syntax is free-form, so a line break is equivalent to a space. An exception to free-formness is a comment, which begins with a semicolon and ends at the end of the line.

Scheme implementations represent lists in linked form, with the empty list represented by a null pointer. The representations of lists (a b c) and (a (b c) ( )) are shown in Figure 11.1.

11.3. Programs and evaluation

A Scheme program is a list, so the syntax of programs is identical to the syntax of lists. A program is just a list that is passed to the Scheme evaluator to be computed. Computation

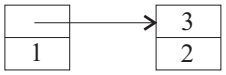
starts by getting (and evaluating) the car of the list. It must be a function or one of the special words such as **if** or **define**, which is performed using the cdr of the list as parameters. For example, when list (+ 2 3) is evaluated, the Scheme evaluator performs function + with parameter list (2 3). The result is 5.

Scheme does not have any special syntactic forms such as binary operators. If you want to compute the product of x and y, write (* x y), not (x * y). The arithmetic functions +, *, − and / all take arbitrarily many parameters, with operations done from left to right. For example, evaluating (+ 3 5 7 9) yields 24.

To avoid confusion between a list and its value as an expression, we write $\langle L \rangle$ for the value produced when list L is evaluated. So, for example, $\langle$(+ 2 3)$\rangle = 5$. We also say that evaluation of (+ 2 3) yields 5. Evaluation is normally inside-out. For example, $\langle$(+ (+ 4 5) (+ 2 6))$\rangle$= 17, with (+ 4 5) and (+ 2 6) evaluated before doing the outer addition. For a few special functions, such as **if**, **cond** and **define**, Scheme deviates from inside-out evaluation, and instead passes the function its arguments as unevaluated lists. The function decides what to do with those lists.

Creating lists in programs

Suppose that you want to build the list (1 2 3). If you write (1 2 3) *as a program*, the Scheme evaluator will presume that you intend the car of this list, the number 1, to be a function. But recall that cons is the Scheme equivalent of the :: operator of Cinnameg. In a program, the empty list is written '( ), and expression (cons 1 (cons 2 (cons 3 '( )))) is the Scheme equivalent of Cinnameg expression 1::2::3::[ ]. If you forget the empty list and evaluate expression (cons 1 (cons 2 3)), you get the a linked list with the following diagram.



That is called an *improper list* in Scheme, since it does not end on ( ) (a null pointer). Be careful with improper lists, since most functions on lists require proper lists. Scheme shows improper list $\langle$(cons 2 3)$\rangle$as (2.3).

You can use function *list* to perform several cons operations in a row. Expression (list 1 2 3) evaluates to the list (1 2 3), and $\langle$(list (+ 1 1) (+ 3 4))$\rangle$ = (2 7). Alternatively, you can suppress evaluation altogether. Expression '(1 2 3) evaluates to the list (1 2 3) and 'X evaluates to exactly X, without evaluation. For example, $\langle$'(the fox hid in the bushes)$\rangle$ is the list of symbols (the fox hid in the bushes). and $\langle$'(+ (+ 2 3) (+ 1 2))$\rangle$ is the list (+ (+ 2 3) (+ 1 2)), a list containing the symbol +, the list (+ 2 3) and the list (+ 1 2).

11.4. Defining functions

To define a function *f*, use **define**. Definition

```
(define (sqr x) (* x x))
```

indicates that (sqr x) is the same value as (* x x), for any x. You can think of this definition as an equation stating that (sqr x) and (* x x) are equal. But there is no equal sign in the syntax. All Scheme constructs use the syntax of lists, without any wasted words or extra symbols.

11.5. Making choices

Scheme does not support definition by cases in the way that we have used in Chapter 9. Instead, a single definition must be augmented by a conditional expression that embodies the choice.

The basic conditional function is **if**. The Scheme system evaluates expression (if a b c) by first evaluating a. If a comes out to be #f (false), then the value of expression (if a b c) is the same as the value of expression c. If a comes out to be anything except #f, then the value of b is taken instead. For example, here is a definition of a function that takes the length of a list. It uses built-in function null?, where evaluating (null? x) yields #t if x is empty and #f otherwise.

```
(define (length x) (if (null? x) 0 (+ 1 (length (cdr x)))))
```

Typically, Scheme programs are presented in multiple lines, with indentation showing structure. The length function would normally be written as follows to improve readability.

```
(define (length x)
  (if (null? x)
    0
    (+ 1 (length (cdr x)))
  )
)
```

It is straightforward to translate simple equational function definitions from Cinnameg into Scheme. Just remember that Scheme has no pattern matching (of the form that we have been using) so you must break things down using basic functions such as car and cdr. Also remember that Scheme has no definition by cases (again, of the form that we have used) so all definitions must be written as a single equation, using conditional functions to make choices. Figure 11.2 shows definitions of a list concatenation function *append* in Cinnameg and Scheme. Append is one of the standard functions provided by Scheme.

Multiway choices: cond

There is a more versatile conditional function called **cond**. Expression (cond (t_1 v_1) (t_2 v_2) ... (else v_n)) evaluates conditions t_1, t_2, etc. until it finds one (say, t_i) that yields a value other than #f. It then yields the value of v_i. If all of the tests yield #f, then it yields v_n. Reserved word **else** is equivalent, in the context of a cond expression, to #t.

For example, here is a definition of (drop n L), which yields the result of removing the first n members of list L, or ( ) if L has fewer than n members. It uses function =, which does an equality test.

```
(define (drop n L)
  (cond
    ((null? L)  '())
    ((= n 0)    L)
    (else       (drop (- n 1) (cdr L)))
  )
)
```

```
{case append ([], y)    = y
 case append (h::t, y) = h :: append(t,y)
}
```

(a)

```
{append (x,y) =
  If* x == []
    then y
    else (head x) :: append(tail x, y)
  %If*
}
```

(b)

```
(define (append x y)
  (if (null? x) y
    (cons (car x) (append (cdr x) y))
  )
)
```

(c)

Figure 11.2: **Definitions of list concatenation (called append here) in (a) Cinnameg (using definition by cases), (b) Cinnameg (with an *if* to handle the cases) and (c) Scheme.** The append function is standard in Scheme.

```
(define (flatten x)
  (cond
    ((null? x) '())
    ((list? x) (append (flatten (car x)) (flatten (cdr x))))
    (else (list x))
  )
)
```

Figure 11.3: **A definition of function flatten, which removes internal structure from a list.** Evaluating expression (flatten '((a b) '( ) ((c) (d)) e)) produces list (a b c d e), where all list structure has been removed, and only the non-list components are left.

Boolean operators

Boolean functions include **not**, **or** and **and**. For example, ⟨(not #t)⟩ = #f, and ⟨(or #f #t)⟩ = #t. The **and** and **or** symbols are not ordinary functions; their evaluation is short-circuited, and they only evaluate enough of their parameters to determine the answer. For example, evaluation of (or #t E) will not evaluate E at all.

Scheme takes full advantage of its typeless nature. All of the standard boolean functions allow parameters that are neither #t nor #f, with anything other than #f treated as a true value. The **or** function, for example, produces the first of its parameters that evaluates to a value other than #f, or returns #f if there are no such values. So ⟨(or '(3 4) 6)⟩ = (3 4).

11.6. Run-time type checking and typeless programming

In Scheme, all type checking is done at run time. Occasionally, you need to check that the data in hand is of the correct type. A *predicate* is a function that returns a boolean result, and Scheme provides predicates that test for membership in specific "types". Typically, predicates have names that end on a question mark. For example, predicate **number?** is true of any number, **boolean?** is true just when its parameter is either #t or #f, and predicate **list?** is true of any list. Other standard predicates include **pair?** (true when its parameter is a nonempty list), **symbol?** (true when its parameter is a symbol), and **procedure?** (true when its parameter is a function, called a procedure in Scheme).

Run-time type checking is versatile, since it allows the program to respond dynamically to whatever kind if data it sees. The philosophy is to get out of the programmer's way and let the programmer do whatever he or she wants to do. To illustrate, Figure 11.3 *flattens* a list, removing all internal structure from it.

The down side of run-time type checking is that almost anything that you write, no matter how full of nonsense, will compile. The main requirement is that the parentheses balance. That forces the programmer to discover mistakes via debugging that would have been pointed out by a compiler that performed reasonable type checking.

```
(define (eval-expr E)
  (cond
    ((number? E) E)
    ((eqv? (car E) 'plus)  (apply + (eval-params (cdr E))))
    ((eqv? (car E) 'times) (apply * (eval-params (cdr E))))
    (else '())             ; confused - return ()
  )
)

(define (eval-params E)
  (if (null? E) '()
    (cons (eval-expr (car E)) (eval-params (cdr E)))
  )
)
```

Figure 11.4: **Definition of an evaluator for simple arithmetic expressions.** This evaluator allows only plus and times and numbers in expressions. Expression (apply f args) has the effect of running function f on the parameters in list args. For example, (apply + (list 2 3)) is treated like (+ 2 3). Predicate eqv? is used to perform equality tests of symbols, as opposed to =, which works for numbers.

11.7. Programs as data: a simple interpreter

Since a program has the same form as a list, it is very easy to treat programs as data and data as programs. One of the built-in Scheme functionsis the Scheme interpreter itself. Evaluating expression (eval e v) runs the Scheme interpreter on list $\langle e \rangle$. That is, it feeds the result of e back into the interpreter to be evaluated again. Using eval, a program can construct another program on the fly and then run it immediately. (The second parameter of eval provides bindings of identifiers to be used in doing the evaluation. Expression (eval e (interaction-environment)) evaluates Scheme expression $\langle e \rangle$ in the same environment as would be used for interaction with a user, which includes definitions made by the current program.)

Since programs and data have the same form, it is easy to write your own interpreter. An interpreter is also called an *evaluator*, since it evaluates an expression. Figure 11.4 shows an evaluator called eval-expr that only allows two functions, plus (addition) and times (multiplication), and that also allows numeric constants. For example, (eval-expr '(plus (times 2 3) 5)) yields value 11. Scheme allows the addition and multiplication functions to have an arbitrary number of parameters. For example, (+ 5 2 8 4) evaluates to 19. Our evaluator also allows an arbitrary number of parameters. The cases in the definition of eval-expr are as follows.

1. If n is a number then (eval-expr n) should just produce n. There is nothing to evaluate.

2. Suppose E is the list (plus a b c d). Then (eval-expr E) must first evaluate each of a, b, c and d, since they might be complex expressions. Function eval-params will do that job. The cdr of E is (a b c d), and (eval-params (cdr E)) should yield list (a', b', c', d') where a' is $\langle$(eval-expr a)$\rangle$, b' is $\langle$(eval-expr b)$\rangle$, etc.

   Since the example expression E requests an addition, the result of (eval-expr E) is

```
(define (map f L)
  (if (null? L) '()
    (cons (f (car L)) (map f (cdr L)))
  )
)
```

Figure 11.5: **A definition of map in Scheme.** This definition is less general than the standard Scheme map function.

the same as the result of $(+\ a'\ b'\ c'\ d')$, where + is the built-in Scheme addition function. Using eval-params, we have computed the list $r = (a'\ b'\ c'\ d')$. Unfortunately, evaluating $(+\ r)$ yields an error. The addition function cannot take a list of numbers as a parameter; the parameters must be in the same list that contains the + function itself. That problem is overcome by using the built-in function **apply**, where ⟨(apply f (list a b c))⟩ produces the same thing as ⟨(f a b c)⟩. So if E is a list whose car is the symbol plus, then ⟨(eval-expr E)⟩ is the same as ⟨(apply + (eval-params (cdr E)))⟩.

3. Multiplication is handled similarly to addition.

11.8. Higher order programming in Scheme

You can create an anonymous function using the built-in function constructor **lambda**, where, for example, the value of expression (lambda (x) (* x x)) is the function that returns the square of its argument. Function (lambda (a b) (list a b)) takes two parameters and produces a list containing those values.

Scheme is a higher order language, with functions as first class values. The **map** function is standard. If expression L evaluates to list (a b c), then expression (map f L) evaluates to the same thing as expression (list (f a) (f b) (f c)). Figure 11.5 show a definition of map in Scheme. Expression (map (lambda (x) (* x x)) (list 3 4 5)) evaluates to list (9 16 25). A little thought shows that function eval-params of Figure 11.4 can be expressed easily using map. Doing so is left as an exercise.

The standard Scheme **map** function is actually a little more versatile than the one written above. It allows any number of parameters, where the first is a function and the rest are lists. If there are two or more lists, then they must have the same length. Evaluating (map f (list a b c) (list u v w)) yields the same thing as evaluating (list (f a u) (f b v) (f c w)). It applies f to the two first members of the lists, then to the two second members, etc.

11.9. Giving things names

Naming using lambda

Cinnameg expression (z + z — ! z = 5 * 5.) evaluates 5*5 and gives a name, z, to the result. But suppose that you could not use **!** to bind z, and had to accomplish the same thing by another mechanism. You could use expression ((z |-> z + z)(5*5)), letting the parameter passing bind z to the result of evaluating 5*5. You can do the same thing in Scheme; ((lambda (z) (+ z z)) (* 5 5)) evaluates to 50.

Naming using let, let* and letrec

The need to use function application to simulate name binding is unpleasant, and Scheme provides a better approach. Expression (let ((*x v*)) *e*) indicates that identifier *x* should be bound to the value of expression *v* and then expression *e* should be evaluated. The value of *e* is used as the value of this let-expression. For example, Scheme expression

```
(let ((z (* 5 5)))
  (+ z z)
)
```

has value 50. Expression (let ((*x v*)) *e*) just abbreviates for ((lambda (*x*) *e*) *v*).

A let expression allows you to bind several names, by providing more than one (variable value) pair in the binding list. For example, expression

```
(let ((x u)
      (y v))
  e
)
```

abbreviates ((lambda (*x y*) *e*) *u v*). But you need to be careful with this. Translating expression (let ((*x* 2) (*y* (+ *x* 1))) (* *x y*)), where the binding of *y* uses *x*, into its lambda form yields ((lambda (*x y*) (* *x y*)) 2 (+ *x* 1)). But that is a problem. The value of *x* is not known outside of the function, but it is used in expression (+ *x* 1). The bindings are only in effect inside the expression that gives the value of the let-expression, not in expressions that perform other bindings.

Scheme provides a way around the scope problem of **let**. Expression (let* ((*x u*) (*y v*)) *e*) abbreviates

```
(let ((x u))
  (let ((y v))
    e
  )
)
```

so that each binding can use previous bindings.

There is another difficulty with let-expressions; they cannot be used to define recursive functions, since the identifier being defined is not available at the point where you would want to use it. Let*-expressions do not help, since they only allow each binding to use prior bindings, not itself. For local recursive definitions, you can use **letrec**. Expression

```
(letrec ((len
          (lambda (x)
            (if (null? x) 0 (+ 1 (len (cdr x))))
          )
         ))
  (len (list 20 30 40)))
)
```

evaluates to 3.

```
(define (sum L)
  (do ( (x     L (cdr x))
        (total 0 (+ total (car x)))
    ((null? x) total)
  )
)
```

Figure 11.6: **A function that computes the sum of the members of a list of numbers using a loop.**

11.10. Imperative programming in Scheme

Although functional programming is encouraged in Scheme, Scheme is not a purely functional language. It also has some imperative aspects.

You can change the binding of an identifier using set!. For example, (set! 'goose '( )) rebinds symbol goose to ( ). Notice that goose needs to be quoted. The arguments to set! are evaluated before being passed to set!. If you write (set! goose '( )), then the value of variable goose is passed to set! as the name of the symbol to be rebound. For that to make sense, goose would need to hold a symbol.

Expression (begin e_1 e_2 ... e_n) evaluates each of e_1, e_2, ..., e_n, in that order, and then produces the value of e_n as its own value. Typically, expressions e_1, ..., e_{n-1} perform actions, using features like set!, and e_n computes a value.

Loops

Scheme offers a kind of loop called a *do loop*. In simplified form it is as follows.

```
(do ((v1 i1 s1) (v2 i2 s2) ...)
  (test r)
  c1 ... cn
)
```

Each of v_1, v_2, ... is a control variable for the loop. Expression i_1 gives the initial value of v_1, i_2 is the initial value of v_2, etc. Each time around the loop expression *test* is evaluated. If the test expression yields #t, then the loop is ended and the value of the loop expression is the value of expression r. If the test yields #f then each of commands c_1, ..., c_n is evaluated, in sequence. (It is allowed for n to be 0, so there aren't any commands to do.) After the loop body is done, each control variable is updated, with v_1 set to value of expression s_1, v_2 set to s_2, etc., where the updates are done in parallel, as for a let-expression. The loop then runs for another iteration, with expression *test* being tried again, and so on. For example, function sum of Figure 11.6 computes the sum of the members of a list of numbers using a loop.

11.11. Exercises

11.1. What is the value of Scheme expression (car (cdr (cons 'horse (cons 'zebra '( ))))?

11.2. Write a Scheme definition of function **member?**, where (member? x L) is #t when x is a member of list L, and is #f otherwise. For this function you need to do equality tests. Use function equal? to do those tests. (Scheme provides four different equality tests, =, eq?, eqv? and equal?, with slightly different meanings. See the Scheme report for the distinction.)

11.3. Write a definition of function **member?** from the preceding exercise in an imperative form, using a do-loop.

11.4. Write a Scheme definition of function **member**, where (member x L) is the first suffix of list L that has x as its car, and is #f if x does not occur in L. For example, $\langle$(member 3 (list 2 3 4 5))$\rangle$ = (3,4,5). Use equal? for equality tests. This is what the standard Scheme member function does.

11.5. Write a Scheme definition of function **prefix**, where (prefix x y) is #t if list x is a prefix of list y, and is false otherwise. The empty list is a prefix of every list, and a list is considered to be a prefix of itself. Use equal? to do equality tests.

11.6. Rewrite **prefix** in an imperative form using a do-loop.

11.7. Write a definition of function **list-equal?**, which compares two lists by comparing their structures. Assume that non-lists can be compared using function eqv?. Be careful to handle list members that are themselves lists using your list-equal? function. For example, evaluating (list-equal? '(a b) (c)) '((a b) (c))) should yield #t.

11.8. Write a Scheme function **descending?** so that (descending? L) is true for a list of numbers L if L is in descending order. Scheme expression ($<$ x y) yields true if $x < y$. The empty list is in descending order by definition, as is a singleton list.

11.9. Write a definition of the foldRtoL function from Chapter 10 in Scheme.

11.10. Function eval-params of Figure 11.4 is really performing a map operation. Show how to replace it in the definition of eval-expr with a use of the built-in Scheme map function.

11.11. Our eval-expr function makes no provision for variables. Write function eval-expr-with-env, where (eval-expr-with-env e B) evaluates expression e in environment B that gives values of variables. B is an *association list.* which is a list of pairs. For example, list ((x 4) (y 12)) indicates that x has value 4 and y has value 12. You will find standard Scheme function **assoc**, described in Exercise 11.16, to be useful in looking up bindings in the environment. Use inside-out evaluation. So if pair (x v) occurs in the environment, then v should be a number, not an unevaluated expression.

11.12. Extend eval-expr-with-env from the Exercise 11.11 to allow an expression of the form (let (x v) e), which evaluates expression e with x equal to the value of expression v. Just add pair (x v') to the environment, where v' is the result of evaluating v in the current environment. You might find the functions mentioned in Exercise 11.15 useful.

11.13. The eval-expr-with-env function of Exercise 11.12 uses eager (inside-out) evaluation, storing a fully computed number with each name. What would be involved in using lazy evaluation instead? Would it be enough simply not to evaluate an expression, and to store the unevaluated expression in the environment, or is more than that required? Think about some examples.

11.14. Scheme supports functions set-car! and set-cdr!, where (set-car! x v) changes the car of list x to be v, and (set-cdr! x v) changes the cdr of list x to be v. The existence of those functions has a bearing on the semantics of lists in Scheme. Argue that it is not adequate to say that a list is a sequence of things. What is a viable semantics of lists in Scheme? Why do you think Cinnameg does not support analogs of set-car! and set-cdr!?

11.15. Scheme provides some abbreviations for extracting parts of lists. Expression (cadr L) is equivalent to (car (cdr L)), and (cdar L) is equivalent to (cdr (car L)). You can have up to 6 letters between c and r. For example, (caddr L) is equivalent to (car (cdr (cdr L))). Say which of these functions should be used for each of the following jobs.

(a) Get the second member of a list.

(b) Get the third member of a list.

(c) Get the first member of the second member of a list. (That is, get x from L, where L is (a (x ...) ...)

11.16. The Scheme function **assoc** is defined so that (assoc x L) yields the first member of list L whose car is equal to x (using equality function equal?). If there is no such member of L, then assoc yields #f.

(a) Assoc is typically used for getting bindings out of lists. For example, (assoc 'x '((y 2) (x 5) (z 9))) evaluates to (x 5), telling you that symbol x is bound to 5. How can you extract the binding of x from the result? Can you use one of the functions described in Exercise 11.15?

(b) You can also use cons cells (improper lists) for the members of L. Evaluation of (assoc 'x '((y.2) (x.5) (z.9))) yields (x.5). How can the binding of symbol x be extracted from the result?

(c) Explain why a programmer might want to use improper lists as the members of list L instead of (proper) lists of length two.

11.17. Write a Scheme definition of the assoc function described in Exercise 11.16.

11.12. Bibliographic notes

Scheme has been revised over time, and is described in various revisions of the Scheme report [80, 81]. Abelson and Sussman [4] use Scheme as an introductory programming language. The Scheme report is available from www.schemers.org/Documents/Standards/R5RS/HTML/. An implementation of Scheme is available from www.drscheme.org.

Steele [91] and Graham [44] describe Common Lisp.

Chapter 12

Types

12.1. Type checking

A *type error* is any error where the programmer is using an operation on a type of data to which it does not or should not apply. For example, taking the square root of a string should be a type error. Adding two characters is a type error, since 'a' + 'b' is surely a meaningless concept on characters. A language is *type-safe* if every type error will be detected, either at compile time or at run time. If it is not type-safe, so that some type errors go entirely undetected, say that a language is *type-unsafe*.

A *dynamically typed* language (also called a *typeless* language) does all of its type checking while the program runs. It keeps tags on data items and checks the tags whenever it performs one of its basic operations. Scheme, Prolog and Python are examples of dynamically typed languages. Dynamically typed languages can be type-safe, but you will not learn about a type error until the error occurs during execution of a program, and finding type errors can be a matter of tedious debugging.

A *statically typed* language requires that some type checking be done by a compiler while it translates a program. Information about types can be used to check that the program is sensible from a type standpoint. Static type checking offers the programmer a kind of security whose worth cannot be overestimated, since a type checker can assure the programmer, once type errors are no longer reported, that certain kinds of errors cannot happen in his or her program.

Say that a syntactically correct program is *well typed* if a compiler detects no type errors in it, and is *poorly typed* if the compiler reports a type error. A *strongly statically typed language* is one that is type-safe, and where all type errors are detected at compile time. In a strongly statically typed language, a well typed program cannot encounter run-time errors due to type mismatches, or due to a programmer's faulty reasoning about types. A strongly statically typed language is at the opposite extreme from a dynamically typed language.

An example of a strongly statically typed language is Standard ML. Languages of the ML family grew out of work on automatic theorem provers, and the type system is critical to ensuring the correctness of the theorem provers. Clearly, they require a rock solid type system, without any loopholes or back doors that allow type errors to slip by.

Most programming languages are neither dynamically typed nor strongly statically typed, but are somewhere in between, with some type errors reported at compile time, some

reported at run time, and (possibly) some escaping detection altogether. Any language that is neither dynamically nor strongly statically typed is *weakly statically typed*. C[1] is an example of a weakly statically typed language. In C, variables must be given types explicitly within a program. If you perform an operation on incorrectly typed arguments, you (usually) get a compile error, and cannot run the program. However, there are some rather large loopholes in the type system that allow type errors to occur when a program runs.

C is type-unsafe. The type errors that are not reported at compile time will never be reported, not even when the running program encounters them. An example concerns the treatment of pointers. In C, pointers occur explicitly in programs. If T is a type, then a value of type $T*$ is a pointer to a variable of type T. For example, a pointer of type int* points to, or refers to, a variable of type int. If you have a pointer of type $T*$, you can *cast* it to another type $S*$, telling the compiler to pretend the memory pointed to contains something of type S rather than type T. There is no checking that the conversion is sensible, and it can cause arbitrarily bad type errors at run time.

Java is another weakly statically typed language.[2] It does not allow the serious unchecked type errors that a C program can make, but a Java program needs to perform some type checking while the program is running. Java has a notion of typing that allows, for example, a horse to be considered a special kind of mammal. Suppose that you have a variable m of type Mammal, which, of course, you know holds a mammal. Based on what you think is going on in your program, suppose you reason that m must actually contain a horse, not a giraffe, or some other kind of mammal, and you would like to put this horse into a variable h of type Horse. In Java, you can write

```
h = (Horse) m;
```

It is impossible for the compiler to determine whether you are right about m holding a horse, and so it allows that. At run time, the Java program will check whether you are right, and if not, it will cause the program to fail.[3] If it turns out that you were mistaken, and that m really holds a giraffe, then you, the programmer, have made an error in reasoning about types that went uncaught by the compiler.

12.2. Sources of type information

Type information used by a compiler can come from three sources. The first concerns built-in operations. A compiler might know that the addition operator can either take two integers and produce an integer, or can take two real numbers and produce a real number. Information about built-in things is loaded into the compiler when it starts.

A second source of information is explicit in a program. If a C program contains declaration `int x`, then the compiler knows that variable x holds an integer. A language might require that the types of all variables, and of other identifiers such as those that name functions, be given explicitly in the program. The C *function prototype*

[1] There are several versions of C, some more weakly typed than others. We will assume that C means ANSI C, a standardized version of C that is better than most about type checking, but is still weakly typed.

[2] You will find books about Java that say that Jave is strongly typed. What they generally mean is that type information about variables is required to be written in Java programs. Here, we are less concerned with how type information is conveyed than with the reliability of the type checking and whether some type checking is done while a program runs. We have avoided the term *strongly typed* to avoid confusion.

[3] It will throw an exception. Exception handling is discussed in Chapter 16.

```
fun
    len([]) = 0
  | len(h::t) = 1 + len(t);
```

Figure 12.1: **Definition in SML of a function that computes the length of a list.** Each line is an equation. The first says that the length of the empty list is 0. Expression $h :: t$ stands for the list whose head is h and whose tail is t, so the second line says that the length of a nonempty list one larger than the length of its tail. This style of definition is discussed in Chapter 9.

```
int f(int,int);
```

is a way of telling the compiler that function f takes two integers as parameters and produces an integer result.

The last source of information is inference based on information that is implicit in a program. Suppose that the compiler has already determined that x and y are integers. When the compiler sees expression $x+y$, it can determine that $x+y$ must produce an integer. Larger expression $z * (x + y)$ then clearly involves a second operand that is an integer, and more the compiler can continue to make more inferences from there. All compilers for typed languages do type inference to some degree.

Some languages, including Standard ML (SML) and Cinnameg, carry the idea of type inference to an extreme, requiring type information only when it cannot be inferred by the compiler. Figure 12.1 shows a definition in SML of a function that computes the length of a list. Notice that the definition contains no explicit type information. But the compiler knows that [] is an empty list, and the presence of expression len([]) tells the compiler that len must be a function that takes a parameter that is a list. Similar inferences at other expressions lead the compiler to conclude, based on the way things are used and the context in which they occur, that len is a function that takes a list as a parameter and produces an integer result. Later in this chapter we look in more detail at how an SML compiler performs type inference.

12.3. A semantics and notation for types

We have talked about types, but have not said just what a type is. The simplest explanation is that a type is a set of values. For example, type Boolean is the set {false, true} and type Char is the set of all characters in a particular alphabet. Saying that a type is a set is a kind of *model theory*. In general, a model theory describes something unfamiliar by modeling it by something that is familiar. The type Integer in a programming language is modeled by the set of values $\{\ldots, -2, -1, 0, 1, 2, \ldots\}$, with which you are already familiar.

The viewpoint that a type is a set will serve us well for the remainder of this chapter. But there is more to be said about just what a type is; Section 14.8 examines an alternative viewpoint.

As a starting point, assume that a few basic types, such as Boolean, Char and Integer, are available. Complex types (also called structured types), whose members are (usually) complex data items, can be built up from those.

Lists

In a statically typed language, a list is a sequence of items of the same type. Some languages have a concept of an *array*, which is a list that is stored sequentially in memory, making some operations efficient. We will treat all kinds of lists in a similar way here, regardless of what they are called or how they are represented. A list such as $[3, 5, 7]$ whose members are integers has type "list of integers", and a list such as [[2], [4,5]] whose members are lists of integers has type "list of lists of integers". Lists can typically be of arbitrary length; the length is not part of the type.

Type "list of integers" is written in different ways in different languages. We will adopt the Cinnameg notation [Integer]. In general, $[T]$ is the type "list of T", where T is a type. That looks similar to the notation for writing a list, with one important difference. If x is a data item, then $[x]$ is a list that has just one member, x. But if T is a type, then $[T]$ is the type of *all* lists whose members have type T, regardless of length. List ["mouse","rabbit","capybara"], for example, has type [String], and [[2,4], [5]] has type [[Integer]].

Tuples and Cartesian products

In a statically typed language, a list is a way to group together values of the same type. But you probably also want to be able to group values of different types, and some other mechanism needs to provide for that.

The Cartesian product $S \times T$ of two sets S and T is the set of all ordered pairs (a, b) where a is a member of S and b is a member of T. Cartesian products in mathematics lead to the notion of Cartesian product types, or simply *product types*. Type Integer × Real contains ordered pairs (x, y) where x has type Integer and y has type Real. Larger tuples can be formed from ordered pairs. For example, ordered triples can be handled by letting a triple (a, b, c) be thought of as an abbreviation for $(a, (b, c))$, where the second member is itself an ordered pair. The type of $(2, ('j', 4))$ is Integer×Char×Integer, where × associates to the right.

For a programming language, it is convenient to choose a notation for types that mimics the notation for the members of the type. Since an ordered pair is written (a, b), we will write the Cartesian product $S \times T$ as (S, T), as is done in Cinnameg. Cartesian product $S \times T \times W$ is written (S, T, W). So (Integer, String, Boolean) is the type of a triple where the first member has type Integer, the second member type String and the third member type Boolean. But, since it is equivalent to Integer×(String×Boolean), type (Integer, String, Boolean) is thought of as a shorthand for (Integer, (String, Boolean)).

Records

You talk about the members of an ordered pair as the left-hand member and the right-hand member. In Cinnameg, for example, there are two functions for getting the members of a pair: $\text{left}((x, y)) = x$ and $\text{right}((x, y)) = y$.

Records are similar to tuples except that their parts are referred to by name rather than by position. The type of a record needs to tell the name and type of each component. We will use notation {shoesize: Integer, height: Real, social: String} to indicate a record type with three components, each having a name and a type. The order of the components in a record is irrelevant. For example, {height: Real, name: String} is exactly the same type

```
{big(x: Integer): Boolean = (x > 100)}
```

Figure 12.2: **Function big, written in Cinnameg, with *domain* Integer and *codomain* Boolean.** Cinnameg allows you to indicate that expression E must have type T by writing $E : T$. Any types that you do not explicitly indicate are determined by the compiler.

```
double max(double x, double y)
{
   if(x > y) return x;
  else return y;
}
```

Figure 12.3: **A C function that takes the maximum of two real numbers.** The type of max is (double, double) → double in our notation for types. (The C language uses its own notation for types. In C notation, max has type double(*)(double,double). We will stick with our notation, regardless of the language being described.)

as {name: String, height: Real}.[4]

Functions

Figure 12.2 show a definition of function big, written in Cinnameg, taking a single integer parameter and producing a boolean result. Each function has a *domain*, the type of the parameter that the function can take, and a *codomain*, the type of the result that it produces. For example, function big has domain Integer and codomain Boolean. If function f has domain type A and codomain type B, then f has type $A \to B$. So big has type Integer → Boolean. If function f has type $A \to B$ and x has type A, then expression $f(x)$ has type B. Expression big(20), for example, has type Boolean (and value false).

Mixing types

Notations for list, product, record and function types are called *type constructors*, since they are used to build types. Type constructors can be combined in arbitrary ways to make more complex types. The addition function (+) for integers has type (Integer, Integer) → Integer. That is, it takes a pair of integers and produces an integer. Since $(4, 7)$ has type (Integer, Integer), expression $(+)(4, 7)$ has type Integer. (Of course, $(+)(4, 7)$ is usually written in the form 4+7, but that is only a syntactically convenient notation, and has no bearing on the meaning.) Figure 12.3 shows function max, written in C, which computes the maximum of two real numbers (type **double** in C). Its type can be expressed as (double, double) → double. Function g defined in Cinnameg by

```
{g(x: Integer, y: Integer) = (x + y, x - y)}
```

takes a pair of integers and produces a pair of integers. It has type (Integer, Integer) → (Integer, Integer).

[4]Cinnameg does not support records in this form, but some other languages, such as SML, do.

What about functions head and tail on lists? Since head($[2, 3, 4]$) = 2, the head function for lists of integers has type [Integer] → Integer. Since tail($[2, 3, 4]$) = $[3, 4]$, it is apparent that the tail function for lists of integers has type [Integer] → [Integer].

12.4. Subtypes

Constrained types

Since we are thinking of a type as a set of values, it makes sense to ask for a subset of a type, or to say that the value of a particular variable is constrained to lie within some given subset. Some languages allow the programmer to specify subsets of types in limited ways. For example, in Ada, if you write

```
X: Integer range 2..5;
```

you indicate that variable X must contain an integer, and that the value of X must be one of the integers 2, 3, 4 or 5. Variable X is said to be of the *subtype* Integer range 2..5 of type Integer.

Subtypes, also called constrained types, are useful for two reasons. One is that the compiler can tell, from the range of values that need to be represented, how much memory a variable needs. For example, a compiler might use only one byte for a variable of type Integer range 2..5, since all of the integers from 2 to 5 can be stored in a byte. An alternative way to obtain the same information (as exemplified by the approach taken by C and Java) is to provide a collection of basic types such as byte, short (two bytes), int (four bytes) etc., indicating the amount of storage needed in machine-specific form. But asking a programmer to tell just which values need to be stored (conceptual information relevant to the programmer) is preferable to asking the programmer to give low level information such as how many bytes are needed for the variable.

A second benefit of constrainted types is error checking. If variable X must lie in the range 2..5, then any attempt to make X hold a value not in that range will be caught (either at run time or compile time) as an error.

Typically, if you write an assignment $X := Y$ in Ada, you expect X and Y to have exactly the same type, or an error is reported by the compiler. With constrained types, it is unrealistic to be so restrictive. Suppose that X has type (Integer range 1..10), and Y has type (Integer range 2..6). Then the assignment is clearly valid, even though the types of X and Y are different. Cases can arise where an assignment might be correct and might not. If, for example, X has type (Integer range 3..7) and Y has type (Integer range 5..10), it might make perfectly good sense to allow assignment $X := Y$. The compiler generates code to check the value of Y.

Subtypes of records

A subtype is a special case. For example, an integer in the range from 1 to 9 is a special case of an integer. Record types also have natural special cases. Suppose that type SHS is defined to be record type {shoesize: Integer, height: Real, social: String} and SH is type {shoesize: Integer, height: Real}. You can argue that a value X of type SHS should be considered to be a special kind of value of type SH, and that the former type is a subtype of the latter, since it has all of the required components (and more). So you should be allowed store X into a variable of type SH. (Notice that you get a subtype of a record type

by *adding* components, not by removing them.) You should also be able to pass X to a function that expects a parameter of type SH. The effect is that functions can work on several different types of parameters. (A function that expects a parameter of type SH can handle any type of parameter that is a record having at least those components.) Generally, a subprogram that works on more than one type of parameter is said to be *polymorphic*. The remainder of this chapter explores polymorphism, and Chapter 19 begins to look at object-oriented programming, which uses a form of polymorphism very similar in spirit to subtypes of records.

12.5. Exercises

12.1. C allows expression 'a' + 'b'. Should that be considered a type error that goes undetected, or should it be considered a normal operation that is not erroneous. Explain.

12.2. Give three ways that a compiler can determine type information.

12.3. List all of the values in the set that models type (Boolean, Boolean).

12.4. Does the Java type **double** have a model-theoretic semantics, or is it too complicated for that? Explain. (In Java, a value of type double is a real number, represented to a given prescribed precision. There is a finite, but large, set of *model numbers* that can be represented.)

12.5. List all of the members of the set that models type Boolean $\rightarrow$ Boolean. What do such things look like, and how can you write one down?

12.6. Does type Integer $\rightarrow$ Integer have a model-theoretic semantics? Explain.

12.7. What is an advantage of record types over Cartesian product types?

12.8. (Requires knowledge of C or C++) C and C++ support records. How would you write type {name: String, size: Integer} in C or C++? Are you required to give the type a name, or can you write it directly as an anonymous record type where it is needed?

12.9. What is the type of each of the following functions? How can you tell? Assume that the addition operation + can only add two integers.

(a) $f(x) = x + x$.

(b) $g(x, y) = \text{head}(x) + y$.

(c) $h(x) = \text{head}(x) + \text{head}(\text{tail}(x))$

(d) $i(x, y) = (y + 1, x + 1)$.

(e) $j(x) = (\text{head}(x), \text{head}(x) + 1)$.

12.10. You can imagine very general constrained types. For example, if T is a type and p is a function of type $T \rightarrow$ Boolean (a *predicate*), it might be possible to create a variable of type (T where p), consisting of all items x of type T such that $p(x)$ is true. Would you advocate allowing an arbitrary predicate to be used as the constraint of a constrained type? Why or why not?

12.11. Java uses four different representations (with four different types) for integers, depending on the number of bytes used to represent the integer. Options are 1, 2, 4 or 8 bytes. What is an advantage of providing four different types of integers? What is a disadvantage as compared with having only one type of integer?

12.6. Bibliographic notes

Pierce [79] covers most aspects of types. Modula-3 [21, 49] is a language with extensive support of subtypes.

Chapter 13

Polymorphism

13.1. Motivation

Dynamically typed languages allow a programmer to write very general functions, such as a function that will take the length of any type of list. Statically typed languages appear to be more restrictive.

Imagine that you would like to provide a general list manipulation facility in an older programming language, such as Pascal. You should provide a type for lists and basic utilities such as a function that computes the length of a list, a function that takes the tail of a list, and so on. You immediately face an almost impenetrable barrier. What kind of values should the lists be allowed to hold? Here is the dilemma.

- If you insist that the members of lists must be integers, then you can provide the facility, but it is almost useless. Chances are that anyone who wants to use lists wants to put things other than integers in his or her lists. Even providing several different types of lists is not adequate, since lists of new types will not be supported. For example, a programmer who creates a new type Widget will not be able to use lists of Widgets.

- On the other hand, if you want to allow arbitrary types of lists, there is no way to express the types of the functions in Pascal, and you cannot write the facility.

Pascal is an example of a *monomorphic* language. Every variable, parameter, etc. must have *one* type. There is no way to express the concept of a general list. For writing simple programs, without significant use of libraries, monomorphic languages such as Pascal are often adequate. But for providing general purpose tools in a statically typed language, some mechanism for dealing with multiple types of items at once is essential. Functions and procedures that can deal with multiple types are said to be *polymorphic*. This chapter explores how polymorphism can be supported and used in a statically typed programming language.

13.2. Ad-hoc polymorphism

Overloading

Pascal has a built-in function called sqr that can be used to compute the square of an integer or the square of a real number. To a certain extent, sqr is polymorphic, since it can be used on more than one type of number. But its polymorphism is very limited. Any time it is used, the compiler knows whether it is squaring an integer or a real number. There would be no fundamental difference in having two functions, one called sqrInt, the other called sqrReal. The only thing offered by function sqr is the convenience that you do not need to remember two function names.

Identifier sqr is *overloaded* in Pascal; that is, it has more than one meaning. Pascal also provides a few other overloaded symbols, such as + and *, which can be used to add or multiply any type of number. Overloading is nothing but a naming convenience, so that it is not necessary to come up with different names for those operations. Overloading is also called *ad-hoc polymorphism*. It is polymorphism that is resolved entirely at compile time; the compiler determines the exact types involved. For example, to write another function called square that is the same as sqr, you need to write two separate definitions of square.

Some languages allow the programmer to overload names. (Pascal does not. It requires that you choose different names for different functions.) But even the ability to do your own overloading is still just a naming convenience. We are looking for something deeper.

Coercion

Suppose that a function called sqrt, of type Real → Real, computes square roots. Obviously, sqrt(2.0) is a correctly typed expression. If constant 2 has type Integer, then expression sqrt(2) would normally be poorly typed. But it is unlikely that sqrt(2) indicates an error in the programmer's reasoning. Rather, the programmer has simply used the mathematical fact that an integer is a special kind of real number. From a mathematical point of view, expression sqrt(2) makes sense, and should probably be allowed.

One way to allow sqrt(2) and similar expressions is to employ a mechanism called *coercion*. A coercion is an implicit type conversion that is inserted by the compiler when it is needed. For example, a coercion rule might state that, if an integer appears where a real number is expected, then the integer is automatically converted, or coerced, to type Real.

Coercion provides an apparent polymorphism that is similar to overloading. It appears that the sqrt function is polymorphic, allowing its parameter to be either an integer or a real number. But the type ascribed to sqrt is still Real → Real, and the polymorphism induced by coercion, while quite useful, is only apparent, and is not what we are looking for if a fully general facility for polymorphism is to be provided.

13.3. Polymorphism and type variables

You are familiar from mathematics with variables that stand for arbitrary numbers and with equations that express properties that such variables are required to have. For example, equation $x + 1 = 2y$ expresses a relationship between x and y. *Type variables* stand for types. We will use Greek letters, such as α and β, for type variables. For example, variable α might stand for Integer or String or any other type. We use equations to express properties

that type variables are required to have. Equation $\alpha = [\beta]$ expresses a relationship between the types that α and β stand for; if β = Integer, then α = [Integer].

When a type involves type variables, we call it a *polymorphic type*. For example, the function that takes the head of a list has polymorphic type $[\alpha] \to \alpha$. So the head function can have type [Integer] → Integer, or [String] → String, or, in general, type $[T] \to T$ for any type T. Some other functions with polymorphic types are as follows.

1. Function tail has polymorphic type $[\alpha] \to [\alpha]$. To see that, look at expression tail(x). Clearly, x must be a list, but there is no constraint on what can be in that list. So x can be assigned type $[\alpha]$. The result of tail(x) must be the same type of list, so it also has type $[\alpha]$.

2. The length function takes an arbitrary kind of list and produces an integer. So its polymorphic type is $[\alpha] \to$ Integer.

3. Notice that 5 :: [3,6] = [5,3,6]. In this example, operator :: takes a pair consisting of an integer (5) and a list of integers ([3,6]), and it produces a list of integers. That is, the :: operator has type (Integer, [Integer]) → [Integer]. But operator :: is much more general than that. Its polymorphic type is $(\alpha, [\alpha]) \to [\alpha]$.

4. Suppose that transpose(x, y) = (y, x). Then transpose has polymorphic type $(\alpha, \beta) \to (\beta, \alpha)$. It would be too restrictive to say that the polymorphic type of transpose $(\alpha, \alpha) \to (\alpha, \alpha)$, because that would say that transpose(3, true) is poorly typed. But transpose$(3, \text{true}) = (\text{true}, 3)$ makes perfectly good sense. Different variables are allowed to stand for different types, but are not required to be different. For example, transpose can work on a pair of integers.

5. Function zip takes a pair of lists and produces a list of pairs. For example, zip$([2, 3, 4]$, ["abc", "def", "ghi"]$) = [(2, \text{"abc"}), (3, \text{"def"}), (4, \text{"ghi"})]$. The example shows zip being used with type ([Integer], [String]) → [(Integer, String)]. But zip can be used with any types of lists. The polymorphic type of zip is $([\alpha], [\beta]) \to [(\alpha, \beta)]$.

Since Greek letters are not available, Cinnameg uses notation $\langle x \rangle$ for a type variable called x, and, in general, allows a type variable to have any name, such as ⟨table⟩. The type of head in Cinnameg notation is $[\langle a \rangle] \to \langle a \rangle$.

13.4. Type checking with type variables

To find the polymorphic type of a function by hand, a good starting place is an example. After writing down the value that the function takes in and the value that it produces, try writing the domain and codomain types of the example. Then generalize, replacing types with variables. Choose an example that is as general as possible, using different types of things wherever possible.

But to have a compiler perform type checking with type variables, we need to have an algorithm that does it. We will start with an algorithm that does not exploit polymorphism to its fullest extent, and then see how to modify the algorithm to allow more polymorphism.

The elementary algorithm breaks the process into three phases.

1. Assign a different type variable to each identifier, constant and subexpression of the definition. In cases where it is clear that two expressions must have the same type, you

are free to choose the same variable, but the algorithm normally just uses a different variable for each subexpression.

2. From the definition, determine equations that must be true in order for the definition to make any sense. Rules for determining the equations are discussed below.

3. Solve the equations using a tool called unification, which will find values for some of the type variables but might leave other type variables unknown. The unbound type variables represent the residual polymorphism in the function. If the equations have no solution then the definition is poorly typed.

It is best to introduce those three phases as they apply to an example. We will use the following function, len, that computes the length of a list.

$$\texttt{len}([\,]) = 0 \tag{13.4.1}$$
$$\texttt{len}(h :: t) = 1 + \texttt{len}(t) \tag{13.4.2}$$

Presume, for simplicity, that + has type (Integer, Integer) → Integer and that 0 has type Integer.

13.4.1. Assigning initial types

Phase 1 for len assigns a type variable to each subexpression that occurs in the definition of len, including len itself. Start with the identifiers and constants. In cases where the types are known (such as for ::, which has a known polymorphic type) attach the known type, but use different variables for each occurrence of the identifier or constant. Notation $A : T$ indicates that A has type T.

$$\begin{aligned} 0 &: \text{Integer} \\ 1 &: \text{Integer} \\ \text{len} &: \alpha_1 \\ h &: \alpha_2 \\ t &: \alpha_3 \\ [\,] &: [\alpha_4] \\ :: &: (\alpha_5, [\alpha_5]) \to \alpha_5 \\ + &: (\text{Integer}, \text{Integer}) \to \text{Integer} \end{aligned}$$

Additional subexpressions that occur in the definition of len are as follows. Expression $h :: t$ is written $:: (h, t)$, applying the :: function to ordered pair (h, t).

$$\text{len}([\,]) : \alpha_6 \tag{13.4.3}$$
$$(h, t) : \alpha_7 \tag{13.4.4}$$
$$:: (h, t) : \alpha_8 \tag{13.4.5}$$
$$\text{len}(:: (h, t)) : \alpha_9 \tag{13.4.6}$$
$$\text{len}(t) : \alpha_{10} \tag{13.4.7}$$
$$(1, \text{len}(t)) : \alpha_{11} \tag{13.4.8}$$
$$+(1, \text{len}(t)) : \alpha_{12} \tag{13.4.9}$$

13.4.2. Deriving equations

Phase 2 involves examining the subexpressions of the definition. Consider expression len([ ]), which occurs in Equation 13.4.1, and that has been assigned type α_6 by (13.4.3). The parameter [ ] of len in this expression has been assigned type $[\alpha_4]$, and len must produce a result whose type is α_6, since that is the type assigned to len([ ]). For this to make sense it should be the case that the type α_1 of len is the same as type $[\alpha_4] \to \alpha_6$. So write down the following equation.

$$\alpha_1 = [\alpha_4] \to \alpha_6$$

General rules for deriving type equations depend on the forms of expressions that a given programming language supports, but a few rules are easy to come up with.

1. Suppose that $f(a)$ is an expression that calls for applying function f to parameter a. Suppose that f has been assigned type α_f, a has been assigned type α_a and expression $f(a)$ has been assigned type α_e. Add equation $\alpha_f = \alpha_a \to \alpha_e$ to the collection of equations.

2. Suppose that expression (a, b) has been assigned type α_e, where a has type α_a and b has type α_b. Add equation $\alpha_e = (\alpha_a, \alpha_b)$.

3. Suppose that expression `(if` a `then` b `else` c`)` has been assigned type α_e, where a has type α_a, b has type α_b and c has type α_c. Then add equations $\alpha_e = \alpha_b = \alpha_c$ and $\alpha_a =$ Boolean.

Additionally, if a definition contains equation $A = B$, then the types of A and B must be the same. Inspection of each of the subexpressions and equations in the definition of len yields the following set of equations, each indicating the subexpression or equation that leads to it.

$$\begin{array}{rcll}
\alpha_1 & = & [\alpha_4] \to \alpha_6 & \text{(from (13.4.3))} \\
\alpha_7 & = & (\alpha_2, \alpha_3) & \text{(from (13.4.4))} \\
(\alpha_5, [\alpha_5]) \to \alpha_5 & = & \alpha_7 \to \alpha_8 & \text{(from (13.4.5))} \\
\alpha_1 & = & \alpha_8 \to \alpha_9 & \text{(from (13.4.6))} \\
\alpha_1 & = & \alpha_3 \to \alpha_{10} & \text{(from (13.4.7))} \\
\alpha_{11} & = & (\text{Integer}, \alpha_{10}) & \text{(from (13.4.8))} \\
\alpha_{11} \to \alpha_{12} & = & (\text{Integer}, \text{Integer}) \to \text{Integer} & \text{(from (13.4.9))} \\
\alpha_6 & = & \text{Integer} & \text{(from (13.4.1))} \\
\alpha_9 & = & \alpha_{12} & \text{(from (13.4.2))}
\end{array} \tag{13.4.10}$$

13.4.3. Solving the equations

A *substitution* is a special case of an equation where the left-hand side is just a variable. A substitution is *proper]* if the variable that occurs on the left-hand side does not also occur on the right-hand side. For example, equation $\alpha = (\beta, \gamma)$ is a proper substitution. A set of substitutions is proper if no variable that occurs on the left-hand side of any of the equations in the set also occurs on the right-hand side of any of the equations in the set. For example, $\{\alpha = \gamma,\ \beta = (\gamma, \gamma)\}$ is a proper set of substitutions since α and β do not occur on any of the right-hand sides.

A proper set of substitutions has a nice property: you can eliminate all of the variables that occur on left-hand sides in the set by replacing each one by its corresponding right-hand

$$
\begin{array}{rcl}
\text{unify}(v, v) & = & \{\} \\
\text{unify}(v, A) & = & \{v = A\} \text{ provided } v \text{ does not occur in } A \\
\text{unify}(A, v) & = & \text{unify}(v, A) \text{ if } A\text{is not a variable} \\
\text{unify}(A \to B, C \to D) & = & S \cup T \text{ where } S = \text{unify}(A, C) \text{ and } T = \text{unify}(S(B), S(D)) \\
\text{unify}((A, B), (C, D)) & = & S \cup T \text{ where } S = \text{unify}(A, C) \text{ and } T = \text{unify}(S(B), S(D)) \\
\text{unify}([A], [B]) & = & \text{unify}(A, B)
\end{array}
$$

Figure 13.1: **Definition of unify**(A, B)**.** If equation $A = B$ has a solution then unify(A, B) yields a proper set of substitutions that is equivalent to equation $A = B$.

In the equations that define unify, v is a variable and A, B, C and D are arbitrary types. The equations should be tried in the order listed, and the first one that matches should be used. If none of these equations gives an answer for unify(A, B) then equation $A = B$ has no solution. That is true of the recursive calls as well. If unify(A,B) calls unify(C,D), and equation $C = D$ has no solution, then $A = B$ also has no solution.

If S is a proper set of substitutions, then $S(A)$ is the result of performing substitutions S on type A. For example, if $S = \{\alpha_1 = \alpha_5, \alpha_3 = (\alpha_5, \alpha_6)\}$ then $S(\alpha_1 \to (\alpha_3, \alpha_5)) = \alpha_5 \to ((\alpha_5, \alpha_6), \alpha_5)$.

side. For example, $\{\alpha = \gamma, \beta = (\gamma, \gamma)\}$ tells you to eliminate variables α and β by replacing each occurrence of α by γ and replacing each occurrence of β by (γ, γ).

Inspection of Equations (13.4.10) shows that the equations can, in general, have arbitrarily complex types on each side, and are not all substitutions. But there is a tool called *unification*[1] that takes an arbitrary type equation and converts it into an equivalent proper set of substitutions (or indicates that there is no solution to the equation). The idea is to convert the first equation into a proper set of substitutions, then use those substitutions to eliminate all variables on their left-hand sides. Now there is one less equation. Repeating the process removes one equation at a time until all of the equations are gone. During the process, we keep track of the type of len, doing required substitutions in its type as substitutions are done. In the end, all that is left is the polymorphic type of len, and that is the answer.

Figure 13.1 defines function unify, where unify(A, B) either returns a proper set of substitutions that is equivalent to equation $A = B$, or indicates that there is no solution.

For example, the first two of Equations 13.4.10 are both already substitutions, and do not involve any common variables. Replacing each occurrence of α_1 by $[\alpha_4] \to \alpha_6$ and each occurrence of α_7 by (α_2, α_3) in all of the other equations (and in the type of len) yields the following remaining equations.

$$
\begin{array}{rcl}
(\alpha_5, [\alpha_5]) \to \alpha_5 & = & (\alpha_2, \alpha_3) \to \alpha_8 \\
[\alpha_4] \to \alpha_6 & = & \alpha_8 \to \alpha_9 \\
[\alpha_4] \to \alpha_6 & = & \alpha_3 \to \alpha_{10} \\
\alpha_{11} & = & (\text{Integer}, \alpha_{10})
\end{array}
$$

[1] Unification is also studied in Chapter 17, on logic programming. Unification for polymorphic types is similar to unification used in logic programming, but the variant that includes the occur-check must be used.

$$\begin{aligned} \alpha_{11} \to \alpha_{12} &= (\text{Integer}, \text{Integer}) \to \text{Integer} \\ \alpha_6 &= \text{Integer} \\ \alpha_9 &= \alpha_{12} \\ \text{len} &: [\alpha_4] \to \alpha_6 \end{aligned}$$

The first equation that is left is not a substitution, but the unification algorithm converts it into set of substitutions $\{\alpha_2 = \alpha_5, \alpha_3 = \alpha_5, \alpha_8 = \alpha_5\}$, and performing those substitutions on the remaining equations yields

$$\begin{aligned} [\alpha_4] \to \alpha_6 &= \alpha_5 \to \alpha_9 \\ [\alpha_4] \to \alpha_6 &= \alpha_5 \to \alpha_{10} \\ \alpha_{11} &= (\text{Integer}, \alpha_{10}) \\ \alpha_{11} \to \alpha_{12} &= (\text{Integer}, \text{Integer}) \to \text{Integer} \\ \alpha_6 &= \text{Integer} \\ \alpha_9 &= \alpha_{12} \\ \text{len} &: [\alpha_4] \to \alpha_6 \end{aligned}$$

This process continues until all of the equations have been eliminated and all that is left is the line indicating the polymorphic type of len. You might end up with len: $[\alpha_5] \to$ Integer. The only thing that can be different in your solution is the name of the variable α_5. But what you choose to call a variable does not matter. (Which variable you end up with depends on the details of how the unification algorithm chooses substitutions. When unifying two variables α and β, you can either replace α by β or replace β by α.)

13.4.4. Another sample type analysis

Define

$$f_1(x, y) = (\text{head}(x), \text{head}(y)). \tag{13.4.11}$$

A type analysis (or type *inference*) of the definition of f_1 starts by choosing type variables for the subexpressions. There is an important point, though, that comes up here. Identifier head occurs twice. But it is a polymorphic function, and there is no reason (at least to start) to think that the two occurrences of head must be used with the same type. So each occurrence is given a type that is a copy, with different type variables, of the polymorphic type of head. To distinguish the two, we write head_1 for the first occurrence of head and head_2 for the second occurrence. Since we do not know in advance the types of parameters or of the function f_1 being defined, we have no polymorphic types to copy, so we use the same type variable for every occurrence of them.

$$f_1 : \alpha_1 \tag{13.4.12}$$
$$x : \alpha_2 \tag{13.4.13}$$
$$y : \alpha_3 \tag{13.4.14}$$
$$(x, y) : \alpha_4 \tag{13.4.15}$$
$$f_1(x, y) : \alpha_5 \tag{13.4.16}$$
$$\text{head}_1 : [\alpha_6] \to \alpha_6 \tag{13.4.17}$$
$$\text{head}_1(x) : \alpha_7 \tag{13.4.18}$$

$$\begin{aligned} \text{head}_2 &: [\alpha_8] \to \alpha_8 && (13.4.19)\\ \text{head}_2(y) &: \alpha_9 && (13.4.20)\\ (\text{head}_1(x), \text{head}_2(y)) &: \alpha_{10} && (13.4.21) \end{aligned}$$

The equations are as follows.

$$\begin{aligned} \alpha_4 &= (\alpha_2, \alpha_3) && \text{(from (13.4.15))}\\ \alpha_1 &= \alpha_4 \to \alpha_5 && \text{(from (13.4.16))}\\ [\alpha_6] \to \alpha_6 &= \alpha_2 \to \alpha_7 && \text{(from (13.4.18))}\\ [\alpha_8] \to \alpha_8 &= \alpha_3 \to \alpha_9 && \text{(from (13.4.20))}\\ \alpha_{10} &= (\alpha_7, \alpha_9) && \text{(from (13.4.21))}\\ \alpha_5 &= \alpha_{10} && \text{(from (13.4.11))} \end{aligned}$$

The first two equations are already substitutions. Eliminating them yields the following.

$$\begin{aligned} [\alpha_6] \to \alpha_6 &= \alpha_2 \to \alpha_7\\ [\alpha_8] \to \alpha_8 &= \alpha_3 \to \alpha_9\\ \alpha_{10} &= (\alpha_7, \alpha_9)\\ \alpha_5 &= \alpha_{10}\\ f_1 &: (\alpha_2, \alpha_3) \to \alpha_5 \end{aligned}$$

Unification can convert the first two of the remaining equations into substitutions $\{\alpha_2 = [\alpha_6]$, $\alpha_7 = \alpha_6$, $\alpha_3 = [\alpha_8]$, $\alpha_9 = \alpha_8\}$, and performing those substitutions yields the following.

$$\begin{aligned} \alpha_{10} &= (\alpha_6, \alpha_8)\\ \alpha_5 &= \alpha_{10}\\ f_1 &: ([\alpha_6], [\alpha_8]) \to \alpha_5 \end{aligned}$$

Finally, replacing both α_5 and α_{10} by (α_6, α_8) as suggested by the equations yields

$$f_1 : ([\alpha_6], [\alpha_8]) \to (\alpha_6, \alpha_8).$$

Variable names are arbitrary. Choosing simpler names α for α_6 and β for α_8 gives

$$f_1 : ([\alpha], [\beta]) \to (\alpha, \beta).$$

13.4.5. Algorithmic analysis of a higher order function

Define function f as follows.

$$(f\ x)\ y = y\ x \qquad (13.4.22)$$

Figure 13.2 shows phase 1 of a type analysis, assigning type variables to subexpressions. Phase 2 yields a collection of equations, shown in Figure 13.3. In this case, all of the equations are already substitutions, and performing them in sequence replaces α_1 by $\alpha_2 \to ((\alpha_2 \to \alpha_6) \to \alpha_6)$, so that is the polymorphic type of f. There are just two variables in the result, and what they are called is not relevant. Choosing shorter names $\alpha = \alpha_2$ and $\beta = \alpha_6$ yields type $\alpha \to ((\alpha \to \beta) \to \beta)$ for f.

$$
\begin{aligned}
f &: \alpha_1 \\
x &: \alpha_2 \\
y &: \alpha_3 \\
f\ x &: \alpha_4 \\
(f\ x)\ y &: \alpha_5 \\
y\ x &: \alpha_6
\end{aligned}
$$

Figure 13.2: **Assignment of type variables to the subexpressions in definition (13.4.22).**

Subexpression	**Equation**
$f\ x$	$\alpha_1 = \alpha_2 \to \alpha_4$
$(f\ x)\ y$	$\alpha_4 = \alpha_3 \to \alpha_5$
$y\ x$	$\alpha_3 = \alpha_2 \to \alpha_6$
$(f\ x)\ y = y\ x$	$\alpha_5 = \alpha_6$

Figure 13.3: **Derivation of type equations from the subexpressions of definition (13.4.22).**

```
{f(x,y) = (g(g(y))) |
  {g(w) = (x,w)}
}
```
(a)

```
{f = ((x,y) |-> (g(g(y)))) |
  {g = (w |-> (x,w))}
}
```
(b)

```
{g = (x |-> (w |-> (x,w)))}
{f = ((x,y) |->
  ((g x)((g x)(y))))
}
```
(c)

Figure 13.4: Part (a) shows a definition that uses a local polymorphic definition inside it. Identifier g is defined once but used twice, with different types. (The syntax is that of Cinnameg, but the semantics of Cinnameg actually does not allow this.) Part (b) shows the same definition in a more basic form, using function expressions. (x `|->` E is the function h defined by $h(x) = E$.) Part (c) shows the result of removing the definition of g from f, creating two separate definitions. Type inference can now be done on the definition of g first, and then on the definition of f, using the inferred type of g in the same way as predefined functions such as head are used. Each use of g in the definition of f is assigned a different copy of the polymorphic type found for g, making it possible for g to be used with more than one type.

13.4.6. Polymorphic local definitions

We have not yet looked at an example that makes a local polymorphic definition within a function definition. Figure 13.4(a) illustrates one. Notice that g is defined (once) inside the definition of f but used twice, with different types at the two uses. For example, if x and y have type Integer, then the occurrence of g in expression $g(y)$ has type Integer $\rightarrow$ (Integer, Integer). But the other use of g (the first one in $g(g(y))$) takes that value of type (Integer, Integer) and produces a value of type (Integer, (Integer, Integer)). So, for example, $f(2,5) = (2,(2,5))$. Type inference needs to take into account the possibility of using a local definition with more than one type.

A polymorphic local definitions can be dealt with by pulling it out as a separate definition, providing extra parameters that give it access to the local variables that it needs. Figure 13.4 illustrates. Although there are technical details, the idea should be clear enough.

Most languages that support polymorphism with type variables allow local definitions to be polymorphic. Cinnameg does not; it assigns the same type to every occurrence of a locally defined identifier, rather than copying the types. That limits the amount of polymorphism that can be used locally, but significantly simplifies handling of local definitions. You can still create a polymorphic local definition in Cinnameg; just write `/:poly:/` after a left brace, as in the following example. .

```
{/:poly:/ f(x) = (g(g(x))) |
  {/:poly:/ g(y) = (x,y).
}
```

13.4.7. Parameters and let-bound polymorphism

The reason that it makes sense for a definition to be polymorphic is that you can see, from the structure of the definition, exactly what it needs to do, and so you can determine the types of things that it will work on. For example, in definition

$$f(x) = \text{head}(x) :: y$$

you can clearly see that x must be a list from the structure of the definition. It is possible to infer the amount of polymorphism that the definition of f has.

Parameters are different. Their actual values are defined elsewhere, and the definition of a function cannot know how its actual parameters will be defined. As a result, a function cannot know to what extent its actual parameters might be polymorphic. To account for that, it must presume that parameters do not possess polymorphism. Each occurrence of a parameter in a function definition must have the same type variable attached to it, so that the parameter has the same type at each of its occurrences. To illustrate, suppose that you try to make the following definitions in Cinnameg.

```
{g(x)   = (x,x)}
{f(r,y) = r(r(y))}
{h(z)   = f(g,z)}
```

Function f uses its parameter r (a function) in two places. Performing substitutions, it appears that $h(2) = f(g,2) = g(g(2)) = g(2,2) = ((2,2),(2,2))$. But type inference on the definition of f yields type $(\alpha \rightarrow \alpha, \alpha) \rightarrow \alpha$, indicating that the domain and codomain types of r must be the same. Function h passes f the polymorphic function g, of type $\alpha \rightarrow (\alpha, \alpha)$.

Since the domain and codomain of g are not the same, the definition of h is poorly typed. The problem is that the polymorphism inherent in the definition of g cannot be passed through a parameter; only one specific type of g can be used. Notice that the following definition, where g is built into f, is well typed.

```
{g(x) = (x,x)}
{f(y) = g(g(y))}
{h(z) = f(z)}
```

Polymorphism works well for definitions, but does not survive parameter passing. Since definitions are often made using a construct that begins with a word such as **let**, polymorphism with type variables is said to be *let-bound*, or restricted to definitions made explicitly and excluding definitions of parameters that are made implicitly at function calls.

13.4.8. Recursion

Should a recursively defined polymorphic function be allowed to use itself with more than one type? For example, should function f defined by

```
{case f([])  = []
 case f([x]) = x ++ x
 case f(x)   = f(tail x) ++ concat(f([x]))
}
```

be allowed? (Function concat concatenates all members of a list together. For example concat($[a,b,c]$) = a ++ b ++ c.) There is no obvious reason why it shouldn't. Unfortunately, the ideas that we have looked at so far cannot handle that. In fact, if a recursive function can use itself with more than one type, then the type inference problem that a compiler must solve becomes uncomputable,[2] making writing a correct compiler impossible.

Languages that use type inference need to find some way around this problem. One method, used by Cinnameg, is to say that, if a function has its type given explicitly, then references to the same name within the body can have any type that fits the the explicitly given type. If no type has been given, then references within the body must have the same type as the defined function. the polymorphic type onto the programmer. The above definition of f will compile if the program says

```
Expect f:  [[<a>]] -> [<a>].
```

13.5. Putting limits on polymorphism

Assume that a language has two numeric types, Integer and Real. A function that squares a number should probably be polymorphic, so that sqr of an integer produces an integer, but sqr of a real number produces a real number. Initially, a type of $\alpha \to \alpha$ seems appropriate for sqr. Unfortunately, that is too general, since variable α can stand for any type, including String or Boolean, but sqr only works on numbers.

What we need is a way to limit the types that a variable can stand for. Let's assume we have a collection of sets of types that we call *genera*. For example, genus AnyNum might be set {Integer, Real}. Adopting Cinnameg notation for type variables, we will allow a type to have an additional constraint indicating restrictions on what variables can stand for, with the polymorphic type of sqr being ($\langle x \rangle \to \langle x \rangle$ where ($\langle x$: AnyNum$\rangle$))

[2] An uncomputable function cannot be solved on a computer. There does not exist any algorithm for it.

Type inference with restricted variables

Allowing variables to have constraints requires some changes to the unification algorithm employed during type inference. Write $\langle a : G\rangle$ for a variable called a that is constrained to stand for a member of genus G.

1. If two variables $\langle a : G\rangle$ and $\langle b : H\rangle$ must be equal, the compiler needs to find the intersection I of genera G and H, and to replace both $\langle a : G\rangle$ and and $\langle b : H\rangle$ by a new variable, $\langle c : I\rangle$. The intersection is chosen because any type T that can both be the value of $\langle a : G\rangle$ and of $\langle b : H\rangle$ must be in both G and H. If I is an empty set, then there is a type error. You can write the intersection of G and H as $G\&{:}H$ in a Cinnameg program.

2. If $\langle a : G\rangle$ needs to be replaced with a named type T, such as Integer, then the compiler needs to check that T is a member of G; otherwise it is a type error.

3. Generalizing the prior rule, suppose that variable $\langle a : G\rangle$ needs to be replaced with a polymorphic type T, where T contains other type variables. The compiler should ensure that T will be a member of G no matter how variables are replaced later during type inference. That requires a few rules that are illustrated by the following example.

   Genus AnyOrder holds types that support order tests such as $<$ and $>$. Lists are ordered lexicographically, a generalization of alphabetical order of strings. In general, type $[T]$ is in AnyOrder just when T is in Order. Consequently, when it replaces variable $\langle a$: AnyOrder$\rangle$ by $[\langle b\rangle]$, the compiler also creates a new variable $\langle c$: AnyOrder$\rangle$, and adds equation $\langle b\rangle = \langle c\rangle$.

13.6. Defining polymorphic functions

Features such as Expect tell you which operations will be defined, but do not define them. There are two common ways to make definitions for polymorphic functions and procedures.

13.6.1. Ad-hoc polymorphic definitions

One way to make polymorphic definitions is to define each function or procedure separately for each type. For example, genus AnyOrder contains types with order tests ($<$, $>$, etc.) It is not possible to define how order tests work for arbitrary types, since they certainly cannot be done the same way for strings as for integers. Each time you add a new type to AnyOrder, you define how order tests work for that type.

Ad-hoc polymorphic definitions are common in the setting of object-oriented programming, where subprograms defined in an ad-hoc manner are called *virtual* subprograms. Chapter 20 explores this idea further in the context of object-oriented programming.

13.6.2. Parametric polymorphic definitions

Suppose that function weight, of type $\langle$AnyAnimal$\rangle\rightarrow$ Real has already been defined. Now you need to define a new function called big, where an animal is defined to be big if its weight is more than 10. That new function is easily written in terms of the weight function. What is more, it only needs to be written once, and will continue to work, even if another type is added later to genus AnyAnimal. The code that implements big is shared by several

types, and is called a *parametric* definition. Think of it as implicitly parameterized by the type on which it works, so that it can request a definition of weight that works on the same type.

To add a new type to genus AnyOrder, it appears to be necessary to define six operations for that type, one for each of $<$, $>$, $\leq$, $\geq$, $=$ and $\neq$. But that can be reduced substantially by adding one more operation, *compare*, where compare is intended to work according to the following equations.

$$\text{compare}(x, y) = \begin{cases} -1 & \text{if } x < y \\ 0 & \text{if } x = y \\ 1 & \text{if } x > y. \end{cases}$$

When you add a new type to AnyOrder, you only define compare. Functions such as $<$ and $>$ can be defined parametrically in terms of compare, and only need to be written once. For example,

$$x < y = (\text{compare}(x, y) = -1).$$

13.7. Exercises

13.1. Explain why overloading is not usually regarded as genuine polymorphism in a programming language.

13.2. How does coercion lead to apparent polymorphism?

13.3. Section 13.2 indicates that Pascal's sqr function is overloaded, allowing it to square either an integer or a real number. Explain why overloading is a better explanation of sqr than coercion. (Pascal will coerce an integer to a real number, but it will not coerce a real number to an integer.)

13.4. Quite a few different coercions have been employed in programming languages. For example, some languages coerce real numbers to integers (by either truncating after the decimal point or rounding); some coerce strings of digits to numbers, and numbers to strings of digits; and some coerce a character to a string of length one.

Give an argument, with examples, for one of the following two positions. (1) Having a lot of sensible coercions frees programmers from having to think so much about types, allowing him or her to write whatever seems to make sense. (2) Having a large number of coercions can actually put more of a burden on a programmer, forcing him or her to think even more carefully about types, than would be necessary if there were no coercions at all.

13.5. Why is polymorphism important to a statically typed programming language? What feature of modern software development environments is most responsible for the need for polymorphism in programming languages?

13.6. Using type variables, give the most general polymorphic type of each of the following functions.

(a) $f(x) = (x, x)$

(b) $g(x) = (\text{tail}(x), \text{head}(x))$

(c) $h(x) = \text{head}(\text{tail}(x))$

(d) $i(x, y) = (y, x, x)$

(e) $j\ x\ y = \text{head}(x\ y)$

(f) $k\ x\ y = x\ y\ y$

13.7. Perform a type analysis, using type variables, of the definition of cat in Equations (9.8.6).

13.8. Perform a type analysis, using type variables, of the definition of take in Equations (9.8.5). Follow the Cinnameg assumption that, when $n + 1$ is used in a pattern, it has type Integer.

13.9. Perform a type analysis, using type variables, of the definition of member in Equations 9.8.4. When a variable occurs twice in a pattern, as m does in the second equation, an equality test needs to be done, implicitly using operator ==, and the polymorphic type of == is $(\langle \text{a: Eq} \rangle, \langle \text{a: Eq} \rangle) \to \text{Boolean}$. (Genus Eq contains types that support equality tests.) Be sure to take the need for equality tests into account.

13.10. Polymorphism can achieve in a typed language much of what is allowed in a typeless language, but it cannot achieve everything. Argue that the following function f makes sense if you ignore types, but that the definition of f is not well-typed. What is $f(3, 2)$?

$$\begin{aligned} f(0, x) &= x \\ f(n + 1, x) &= (f(n, x), f(n, x)) \end{aligned}$$

13.11. Prove that the unification algorithm described in Figure 13.1 produces a proper set of substitutions.

13.12. Prove that the set of substitutions produced by unify(A,B) are, taken together, equivalent to equation $A = B$.

13.13. How can the algorithm for doing type inference with polymorphic local definitions be extended to handle mutually recursive definitions?

13.14. Is Cinnameg definition {`f(p) = (p(1), p("abc"))`} well-typed? Why or why not? What if f is used as follows?

```
{id(x) = x}
{z = f(id)}
```

13.15. Is the polymorphic definition of function cat given by equations (9.8.6) parametric or ad-hoc?

13.16. Give an example where ad-hoc polymorphic definitions appear to be necessary.

13.17. Suppose that there are only two basic types, Boolean and Integer, and that there are two type constructors, the pair constructor and the list constructor. So, for example, (Integer, [Boolean]) is an allowed type. Using only those, write a sensible definition of a function toString that converts an arbitrary thing to a string. For example,

toString((2,true)) = "(2,true)", and toString([2,4,6]) = "[2,4,6]". Your definition of toString should be able to handle arbitrarily complex types built from those constructors. For example, toString((2,[1,5]), (true,false)) = "((2,[1,5]), (true,false))". Are you using ad-hoc definitions, parametric definitions, or a combination of the two?

13.8. Bibliographic notes

Type inference with type variables was discovered by Hindley [50] and Milner [70], and Cardelli [20] explains type inference. Cardelli and Wegner [19] survey polymorphic types.

Chapter 14

Creating New Types

Fortran IV was designed for scientific computing with real numbers. It provides types DOUBLE PRECISION (real numbers), INTEGER and a few other simple types, plus arrays. The primary purpose of types is to allow the compiler to have enough information to compile programs. For example, real numbers occupied more memory than integers and arithmetic on integers uses different instructions from arithmetic on real numbers. The types that are provided by the language are all that can ever be used. Algol 60, another early programming language, similarly provides a few basic types such as integer, real and character, without offering the possibility of creating new types.

A programmer can often get by without any types except the basic ones. For example, if you need to use a point such as (2.0, 3.0) in the cartesian plane, an array of two real numbers will do the job. But an important feature provided by a programming language is the ability to abstract, or to hide details. The programmer might prefer to use a type Point, rather than a more detailed type such as (Real, Real). The type (Real, Real) is merely the *representation type* of type Point — the type under which points are stored.

This chapter initially adopts the model-theoretic view of a type as a set of values, as discussed in Chapter 12, a viewpoint that will necessitate avoiding operations that change things. Then we will examine a different way to define a type that makes sense for mutable data types.

14.1. Defining data types

Programmers make their products easier to understand by adding new concepts to them. For example, new functions and procedures implement new computational concepts. New types of data are a natural extension of that idea.

We use Cinnameg for illustration. Other programming languages exhibit similar characteristics. In Cinnameg, you create a new type using a type declaration, indicating that a new type is represented by a previously known type. For example

```
Type Distance = distance Real.
```

defines a new type Distance to be represented by type Real. Think of type Real as a set of numbers. For each member of type Real there is a corresponding member of type Distance, which you might understand to represent a given distance, in meters. So, corresponding to the real number 3.4 is the distance (3.4 meters).

Constructors and unconstructors

Type Distance is not the same as Real, in spite of the close correspondence between the two. In order to use Distance, you need a way to convert from Real to Distance and from Distance to Real. When you define Distance as above, the Cinnameg compiler provides two functions for those conversions: a *constructor* called *distance* that converts from Real to Distance, and an *unconstructor* called *undistance* that converts from Distance to Real. So distance(4.0) is (4.0 meters), the member of type Distance that corresponds to real number 4.0, and undistance(distance(4.0)) is 4.0 (a real number).

Rather than using unconstructors directly, you typically use pattern matching to convert from Distance to Real. Pattern match

```
{distance(r) =~d}
```

binds r to the real number that corresponds to distance d.

Defining operations on new types

When you define a new type, only a few operations are provided for your new type, including a constructor and an unconstructor and possibly a few other operations. It is up to you to provide additional operations. For example, if you want to be able to add two distances (producing another distance), you define + for distances.

```
{distance(x) + distance(y) = distance(x+y)}
```

It is usually a good thing that other operations are not defined automatically. For example, you would not want an automatic definition of multiplication, where the product of two distances is a distance, since the product of two distances should be an area. A more sensible thing is to define type Area, and then multiplication of distances as follows.

```
{distance(x) * distance(y) = area(x*y)}
```

14.2. Type checking

It is important to distinguish between creating a genuinely new type and just giving a new name to an existing type. Say that two types A and B are *equivalent* if an item of type A can be used where an item of type B is expected, and vice versa. When you give a new name to an existing type, the new and old types are equivalent. When you create a new type, your new type is not equivalent to its representation type. Giving a new name to an old type is said use the *structural equivalence* rule; the two types are equivalent because their representations have the same structure. Creating a new type uses the *name equivalence* rule: two types are only equivalent if they have the same name or exact written form.

To illustrate the distinction, suppose that you need two data types; type Point represents a point in the cartesian plane, and type Complex represents a complex number. Each is represented by a pair of real numbers. If types Point and Complex just rename type (Real, Real), then they are equivalent. Accidentally passing a Point to a function that requires a value of type Complex will not lead to a type error. But when Point and Complex are new types, the compiler performs much more stringent type checking. Programmers who work in a statically typed language expect type errors to be reported, and name equivalence has become the preferred rule.

Not all programming languages employ just name equivalence or just structural equivalence. In Haskell, definitions

```
type Integer1 = Int
data Integer2 = integer2 Int
```

makes Integer1 another name for type Int, but creates a new type Integer2 whose representation type is Int. Similarly, in C++,

```
typedef Integer1 = int;
struct Integer2 {int n;};
```

creates a new name Integer1 for type int and creates a new type Integer2, not equivalent to int. Pascal provides an unusual example. In Pascal,

```
type MyInt1   = integer;
     MyArray1 = array [1..4] of integer;
     MyArray2 = array [1..4] of integer;
```

makes MyInt1 just a new name for type integer. But types MyArray1 and MyArray2 are inequivalent types. Pascal uses different rules, depending on what the representation type is. Irregular rules such as that are strongly discouraged in language design. Ada, a successor to Pascal, adopts name equivalence in all type definitions.

14.3. Structured types and selectors

When you define a new type that is represented by a structured type, you typically also define operations that select parts of the representation. In Cinnameg, for example, if type Computer is defined by

```
Type Computer = computer (
  manufacturer : String,
  model        : String,
  serial       : Integer)
%Type
```

then it is as if Computer were defined to be represented by a tuple

```
Type Computer = computer (String, String, Integer).
```

but functions manufacturer, model and serial are automatically defined as follows.

```
{manufacturer(computer(m,?,?))  = m}
{model(computer(?,m,?))  = m}
{serial(computer(?,?,s)) = s}
```

If those functions had not been provided, you could (and probably would) write them yourself. That is, for example, how a "type" is defined in a typeless language such as Scheme. If a computer is represented by a list of three quantities, then you could define

```
(define (manufacturer c) (car c))
(define (model c) (cadr c))
(define (serial c) (caddr c))
```

in Scheme, where (car c) is the fist member of list c, (cadr c) is the second member and (caddr c) is the third member.

Selection for list and array types

A list has a first member, a second member, and so on, and the position of a member is called its *index*. Using the idea of an index, you can view a list as a small function, mapping indices to values. For example, list $L = [5, 90, 23]$ implicitly represents the following function, with domain {1, 2, 3}.

x	$L(x)$
1	5
2	90
3	23

So $L(2) = 90$, and in general, the value at index x in list L is the value $L(x)$, treating L as a function. Typically, a list has a small range of integers as its domain, but you can imagine other domains. For example, Ada allows the indices to be any type that is represented by a small range of integers. Ada declaration

```
X: array(Integer range 4..7) of Integer;
```

creates array X so that its domain is {4,5,6,7}. The first member of X is $X(4)$. But Ada also supports *enumerated types*, where the programmer lists the members of a type. A type of allowable traffic light colors can be written as follows in Ada.

```
type TrafficLightColor is (Red, Amber, Green);
```

Type TrafficLightColor has just three possible values, called Red, Amber and Green, and they are represented by small integers, with Red = 0, Amber = 1 and Green = 2. An array that has TrafficLightColor as its domain type, and that holds values of type Integer, is defined as follows in Ada.

```
X: array (TrafficLightColor) of Integer;
```

Now it makes sense to talk about X(Red), X(Amber) and X(Green).

14.4. Choice in data

Sometimes you need a new type whose values have several different kinds, or variants. You can represent such data types by having more than one representation type and more than one constructor. For example, suppose that you need to represent data about different kinds of articles of clothing, with different characteristics for each kind. Figure 14.1 shows a data type for clothing, written in Cinnameg, with three kinds, socks, pants and shirts. Now expression `socks(11)` creates a pair of socks (of type Clothing) with size 11. You can also write `socks(size : 11)`, showing the name of the parameter. If `shortSleeve` is a member of `SleeveType`, then expression `shirt(chest : 40, neck : 15, sleeve : shortSleeve)` creates a shirt, also of type Clothing,

Data types with multiple forms are quite common in software. A program that works on representations of people will probably need to store different kinds of data about people who play different roles. Data about a shipment might depend on the status of the shipment, whether it is in transit, arrived at its destination, etc. A list has two forms, an empty list and a nonempty list. Another example comes up in compiler design. The initial stage of a

```
Type Clothing =
    socks (size   : Integer)
  | pants (waist  : Integer,
           inseam : Integer)
  | shirt (chest  : Integer,
           neck   : Integer,
           sleeve : SleeveType)
%Type
```

Figure 14.1: **A data type for describing articles of clothing, written in Cinnameg.** There is one constructor for each kind of clothing.

```
Type Token =
    identifierToken(String)
  | integerConstantToken(Integer)
  | realConstantToken(Real)
  | ... (more cases)
%Type
```

Figure 14.2: **A type for tokens.**

compiler, called the lexical analyzer, reads individual characters of a program and packages them up into *tokens*. Examples of tokens are a reserved word, such as **if**; an identifier such as **size**; a number such as 24; and a special character such as **(**. The lexical analyzer tells the next stage of the compiler what kind of token it has read, and provides an *attribute* that gives additional information about the token, if necessary. The attribute of an identifier might be the name of the identifier, a string. The attribute of a token such as 334 might be an integer, and the attribute of 2.53 might be a real number. Clearly, the token must have several forms. Figure 14.2 shows a definition of type Token.

Computation on data types with variants

To perform computations on types that involve variants you typically use programs that involve choices that match the variants. For example, suppose that you want to compute the "weft" of a piece of clothing. where the weft of a pair of socks is its size, the weft of a pair of pants is the waist size, and the weft of a shirt is the chest measurement. Figure 14.3(a) shows a definition of weft. Notice that there is one case for each kind of clothing article. Similarly, suppose that you would like to print a value of type Token. You will need a case for each kind of token, as is shown in Figure 14.3(b). Choice in data types is mimicked by choice in computation.

Constants in data types with variants

A shirt has three features, its chest and neck measurements and its kind of sleeve. But some variants of a type can be featureless, having no extra information. For example, suppose that a new article of clothing, a hairpin, is added, and that there is only one size of hairpin,

```
{case weft(socks(size : sz))        = sz
 case weft(pants(waist : w, ?))     = w
 case weft(shirt(chest : c, ?, ?))  = c
}
```

(a)

```
{case PrintToken(identifierToken(name)). =
    Display "Identifier:  %(name)".
 case PrintToken(integerConstantToken(i)). =
    Display "Integer:  %(i)".
 case PrintToken(realConstantToken(r)). =
    Display "Real:  %(r)".
}
```

(b)

Figure 14.3: **Two functions defined using choice.**

so no additional information needs to be stored with one. The definition of Clothing from Figure 14.1 can be modified as shown in Figure 14.4(a), where there is no data associated with the hairpin. Now hairpin is a constant of type Clothing.

When all of the members of a type are featureless the type is called an *enumerated type*. For example, Figure 14.5 defines two enumerated types, ColorType and SleeveType. Again, the obvious way to compute with these things is by cases. Figure 14.5 shows a definition of a function that tells whether a color is "warm".

Implementation of data types with variants

Values with variants are represented using *tags*; think of each item as an ordered pair (tag, v) where tag tells which variant it belongs to and v is the value. The tags are typically integers; for type Token you might use 0 for integerToken, 1 for identifierToken, 2 for integerConstantToken, etc. Value identifierToken("size") would be represented by ordered pair (1, "size"). A value of an enumerated type can be represented by just its tag, so an enumerated type has a representation type that is a (small) range of integers.

Untagged unions

A few languages, such as C, offer an alternative way of creating data types with choice that avoids tags. An token attribute type in C can be written as follows.

```
typedef union AttributeType AttributeType;
union AttributeType {
  string identifierAttr;
  int    integerAttr;
  double realAttr;
};
```

```
Type Clothing =
    hairpin
  | socks (size  : Integer)
  | pants (waist : Integer,
           inseam : Integer)
  | shirt (chest : Integer,
           neck  : Integer,
           sleeve: SleeveType)
%Type
```

(a)

```
{case category(~hairpin) = 0
 else category(?)  = 1
}
```

(b)

Figure 14.4: **A modified clothing type, written in Cinnameg.** Part (a) shows type new type definition and part (b) defines a function for the new type. Notice that hairpin is a constant of type Clothing. Pattern ~hairpin only matches hairpin.

```
Type ColorType = red | orange | yellow | green | blue.
Type SleeveType = shortSleeve | longSleeve.
{warm: ColorType -> Boolean by
 case warm(~red)    = true
 case warm(~orange) = true
 case warm(~yellow) = true
 case warm(~green)  = false
 case warm(~blue)   = false
}
```

Figure 14.5: **Definitions of two enumerated types in Cinnameg,** and definition of a function on colors.

A value of type AttributeType[1] is either a string, an integer or a real number (type double), but there is no indication in the value itself of which kind it is. All of the different kinds of values occupy the same memory, and when you store one, you replace (or invalidate) the others. The programmer must know, by other means, which kind value is stored. An obvious way to do that is to create your own tag, and to define type Token as follows.

```
typedef struct Token Token;
struct Token {
  int tag;
  AttributeType attr;
};
```

The code for a compiler might contain

```
if(token.tag == integerToken) {
  int tokenValue = token.attr.integerAttr;
  ...
}
```

using token.attr.integerAttr to select the integerAttr interpretation of the attribute.

Untagged unions are flexible, but they represent a serious breach of type safety. If a programmer makes a mistake, his or her program can treat a value of one type (for example, string) as if it were the representation of another type (for example, double), with unpredictable results.

14.5. Recursion in data

Recursion is a phenomenon that occurs frequently in data types. A large list contains a smaller list (its tail) as part of it. A large tree contains smaller subtrees. Figure 14.6 shows a definition of a type of binary trees, where a tree is either a leaf that holds a string or is a nonleaf that holds a string and two subtrees, and function treeSize, which gives the number of nodes in a tree. Figure 14.7 shows a tree and how that tree can be created in a program.

Notice two things about the definition of function treeSize. First, there are two kinds of trees, leaves and nonleaves. Corresponding to that, there are two cases in the definition of treeSize. Second, recursion in the definition of treeSize matches recursion in the definition of the BinaryTree type. A leaf is the basis of the recursion in both the type and the function definition. A nonleaf is defined in the type using recursion, and the definition of treeSize has matching recursion. A recursive definition whose choices and recursion matches the form of the type definition is called a definition by *primitive recursion*. Function treeFlatten, which performs an inorder traversal of a tree, converting it into a list of strings, is another example of primitive recursion.

```
{case treeFlatten(leaf(s)) = [s]
 case treeFlatten(tree(s,lft,rgt)) =
    treeFlatten(lft) ++ [s] ++ treeFlatten(rgt)
}
```

[1]The type created in the union is called `union AttributeType` in C. The typedef line allows you to use the shorter name, AttributeType.

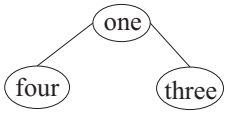
```
Type BinaryTree =
    leaf (leafLabel : String)
  | tree (treeLabel    : String,
          leftSubtree  : BinaryTree,
          rightSubtree : BinaryTree)
%Type
```

(a)

```
{case treeSize(leaf(?))          = 1
 case treeSize(tree(?,lft,rgt)) = 1 + treeSize(lft) + treeSize(rgt)
}
```

(b)

Figure 14.6: **(a) A recursive definition of a type of binary trees and (b) a definition of function treeSize, yielding the number of nodes in a tree.**



(a)

```
tree("one", leaf("four"), leaf("three"))
```

(b)

Figure 14.7: **A tree (a) and an expression whose value is that tree (b) using type BinaryTree of Figure 14.6.**

```
Type BinarySearchTree =
    emptyTree
  | node(key   : String,
         left  : BinarySearchTree,
         right : BinarySearchTree)
%Type
```

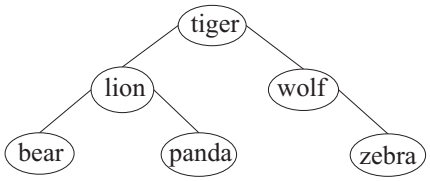
Figure 14.8: **Cinnameg definition of a type for binary search trees, used to represent sets of strings.**

```
{case member?(x, (~emptyTree)) = false
 case member?(x, node(k, lft, rgt))
    = true            when x == k
    = member?(x,lft)  when x < k
    = member?(x,rgt)  when x > k
}
```

Figure 14.9: **A membership tester binary search trees.** (The question mark is part of the name.)

Example: binary search trees

A binary search tree is a kind of tree that is used for organizing sets and tables. It has the characteristic that, for each node N holding value v, all of the nodes in the left subtree of N hold values that are less than v, and all of the nodes in the right subtree of N hold values that are greater than v. For example, the following is a binary search tree that represents the set of strings {"bear", "lion", "panda", "tiger", "wolf", "zebra"}, using alphabetical order.



Type BinaryTree is not quite right for binary search trees, since a binary search tree needs some nodes to have just one subtree. One way to fix that is to introduce an empty tree, letting a leaf be a tree with two empty subtrees. Figure 14.8 shows type BinarySearchTree that uses that idea.

Figure 14.9 shows a definition of a function that tests for membership of a given string in a binary search tree, using the principles of function design. There is one case for the empty tree, but the case for a nonempty tree is broken into three parts for convenience.

A function that inserts a value into a binary search tree can be designed using the same principles. The insert function takes a string x to insert and a tree t in which to do the insertion, and returns the tree that results from inserting x into tree t. Figure 14.10 shows a definition of insert.

```
{case insert(x, (~emptyTree)) = node(x, emptyTree, emptyTree)
 case insert(x, node(k, lft, rgt))
    = node(x, lft, rgt)            when x == k
    = node(k, insert(x,lft), rgt)  when x < k
    = node(k, lft, insert(x,rgt))  when x > k
}
```

Figure 14.10: Insert(x,t) produces the result of inserting x into binary search tree t. This function is consistent with a persistent implementation of binary search trees, the kind of type that we are restricting ourselves to in this chapter. It does not modify the tree, but computes a new tree.

Implementation of recursion in data types

Recursion in types needs to be implemented with care. Typically, a given type of data occupies some fixed number of bytes. If you try to put a tree inside a tree, you find that a tree must occupy more bytes than a tree, which is impossible. To avoid that problem, a recursive use of a data type is usually implemented as a pointer to the actual representation. In languages that support explicit pointers, such as C++, recursion must be done via an explicit pointer.

Data types with recursion typically require variants since otherwise there would be no way to break the recursion. The nonrecursive cases correspond to basis cases in recursive algorithms. Using pointers allows you to take advantage of an implicit variant in the pointers to break recursion; a pointer can either be null or nonnull. A binary tree type can be defined as follows in C++, relying on null pointers to represent termination of the recursion. Type Tree* is the type of a pointer to a tree.

```
struct Tree {string data; Tree* left; Tree* right; };
```

With this definition of type Tree, a leaf is a tree node whose left and right pointers are both null. There is no need to use an explicit union type to create an empty tree, and there are no extra tags to keep track of.

14.6. The formal view of a type

Up to now we have adopted a *model theoretic* viewpoint that a type is a set of values. But an alternative *formal* viewpoint avoids thinking about what the data items are and instead concentrates on what you can do with them. We will start by looking at the formal view of types that can also be described as sets, so that we end up with two consistent definitions. In the final section we turn to types for which model theory fails, and only the formal approach yields a sensible definition.

The signature of a type: type syntax

The first part of the formal view of a data type is the *signature* of the type, which is collection of constants and functions, each with a given type. For example, a natural number is (by our definition) a nonnegative integer. The signature of type Natural might include function +,

which takes two natural numbers and produces a natural number, a constant called 0 that is a natural number, and a few other components. A fairly sparsely populated natural number type would have a signature (0: Natural, 1: Natural, +: (Natural, Natural) → Natural, *: (Natural, Natural) → Natural, ==: (Natural, Natural) → Boolean), consisting of constants 0 and 1, the addition and multiplication operations, and a test to see whether two natural numbers are equal.

Type semantics in the formal view

The signature does not say anything about meaning, just as the syntactic specification of a programming language does not say what programs mean. The second part of the formal view of a type introduces semantics.

In the formal view, there is no model, so we cannot say what 0 stands for, or what it means to add two numbers. Instead, we state some properties that the functions and constants in the signature must have. Here, for example, are some of the relationships that are part of the formal view of the natural-number type.

$$\begin{aligned} x + 0 &= x \\ x + (y + 1) &= (x + y) + 1 \\ x * 0 &= 0 \\ x * 1 &= x \\ x * (y + 1) &= (x * y) + x \\ (0 == 0) &= \text{true} \\ (x + 1 == 0) &= \text{false} \\ (x + 1 == y + 1) &= (x == y) \end{aligned}$$

The formal view is a useful one for reasoning about programs, since it provides, in the properties, the fundamental facts from which to reason. For example, you can use the operations in the signature of Natural to write a definition of a function that exponentiates a number to a positive integer power. (Just write a loop.) Using the properties of the integer operations, and reasoning methods such as the logic of Chapter 24, you can prove that your definition of exponentiation is correct. The formal view gives an *axiomatic semantics* to a type, in the same way that the Hoare logic of Chapter 24 is an axiomatic semantics of a small programming language.

The formal view is useful for specifying the intended semantics of a data type without giving an implementation. Any implementation that has the right signature and satisfies the properties of the type is, by definition, correct.

14.7. Abstract data types and encapsulation

A type that is viewed using the formal view is called an *abstract data type*. The idea is that it only supports the operations given in the signature, and no other operations can be used on it unless they are defined in terms of the ones in the signature. A language that supports abstract data types should enforce the requirement that only operations in the type's signature can be used. That can put some stress on a programming language.

When you define operations on a type, you need to know the type's representation, which includes features that are not part of the signature. It is not possible, for example, to

define the details of how points in the plane work unless you can make use of the representation so that you can extract the components (the x- and y-coordinates). To implement a set of strings as a binary search tree, you need to know details of binary search trees, which certainly are not part of the signature for a set of strings. Typically, you produce one module that implements operations on your new type. In all other modules, you would like to ensure that the representation type is completely hidden; those other modules should be restricted to the type's signature.

A new type is said to be *fully abstract* in modules where all aspects of its representation are hidden, and only operations that are part of the type's signature can be used. Those modules are not allowed to know, for example, that a point is a record that has a component called x, or even that it is a record type at all. A fully abstract type is a kind of encapsulation, and a big advantage of any kind of encapsulation is ease of modification. Changes to the representation of a type cannot affect any modules where that type is fully abstract, as long as the changes respect the signature and semantics.

It is impossible to create a fully abstract type using the structural equivalence rule, since then the representation type is an inherent characteristic of the type. You might expect the name equivalence rule to make representations of types hidden, encapsulated behind the name of the type. But even with name equivalence, you can get some unexpected dependence on representation. For example, C uses name equivalence for structures (records). If you define two structure types

```
typedef struct {double x,y;} Point;
typedef struct {double re,im;} Complex;
```

then, as name equivalence suggests, types Point and Complex are different types, and you cannot pass a value of type Point to a function that expects a parameter of type Complex. But if you write

```
Point p;
p.x = 1.0;
```

then you are relying on the knowledge that a point has a component called x, which you can only know by looking at the representation. Name equivalence *enables* fully abstract types, but is not enough. You must be sure to hide the existence of any operations, such as component selectors, that are specific to the representation.

There can be other operations that also need to be hidden to achieve a fully abstract type. One is assignment. In C, for example, you can perform assignments on structures, as in the following example.

```
Point p,q;
...
p = q;
```

The definition of C says that assignment on structures is done by copying each component from one structure into the other. That is, the definition of the assignment operator is determined by the representation. Sometimes this gives you what you want, and sometimes it does not. If you change the representation later, then you might find that the C compiler's idea of assignment is not what you want, and changing the representation has the effect of changing what assignment does. For example, if you decide to represent the components of a point as a linked list, then the standard definition of = will not cause the list to be copied;

it only copies the pointer to the first cell in the linked list. For a fully abstract type, you need to remove *all* dependence on the representation.

Sometimes even apparently basic operations are not appropriate for new types, and should not be defined automatically. In addition to the assignment operator, compilers sometimes automatically define an equality testing operator for new types. C does that, for example, when the type is represented by a pointer. But an equality operator might not be appropriate for your type; maybe you do not want to allow comparing two Widgets to see whether they are equal. If an equality test is not part of the signature, you do not want it added for you by the compiler.

Ada allows a programmer to create a type whose details are completely hidden within a single module. Definition

```
package Points is
  type Point is private;
  ...
  private
    type Point is record
      X, Y: Float;
    end record
end Points;
```

makes Point available to other modules, but does not define any operations on it other than those explicitly mentioned in the (...) part. Ada packages are broken into interface parts (shown above) and package bodies, where implementations are written. The definition of a type is required to be written in the interface part, as shown, so that the compiler knows how many bytes a value of that type needs, but the compiler reserves that information for itself. Cinnameg uses a similar approach, but does not require mentioning anything about the representation of a type in the interface part of a package. Saying

```
Type Point interface.
```

makes type Point available, but not any functions, or any kinds of operations, on it. Only those operations that are explicitly mentioned in the interface become available to other packages. In the implementation part of the same module, you define the type by giving its representation. The constructor and unconstructor are available only in the implementation part, where they are needed.

14.8. Types without models

Chapter 6 discusses boxes, or first-class variables. A box is an object that can hold one thing, called the content of the box, which can change over time. There are two operations, one that gets the content of the box and another that changes the content. In Cinnameg, a box that holds an integer has type [:Integer:].

In the model-theoretic view, type Boolean has two members, true and false, and those values are created when the type is created. You cannot create new values of type Boolean, or destroy them, while the program runs. Type Integer also has a fixed set of members.

Box types are different. You can create boxes on the fly and later destroy them, so the membership of a box type is constantly changing. Instead of saying that a given box is a member of type [:Integer:], we say that it an *instance* of that type. In general, we will call a type whose instances have time-varying characteristics a *mutable type*.

Semantics of mutable types

The model-theoretic view does not work for mutable types. Since instances can be created and destroyed, there is no fixed collection of things to model. But the formal approach can still work. The signature of a mutable type lists the functions and procedures that are available for that type, just as it does for a type that is a set of values.

The semantics that is associated with the signature needs to be described by axioms, or properties of the procedures and functions that express how procedures change things and characteristics of the results produced by functions. You can give a precondition/postcondition pair for a sequence of operations, similar to what is done in Hoare logic (Chapter 24). Axiom $\{p\}S\{q\}$ states that, if condition p is true, and you perform the sequence of steps S, then condition q will be true afterwards. ($\{\}$ stands for $\{$true$\}$.) For example, in Cinnameg, the signature for type [:Integer:] has function @: [:Integer:] $\rightarrow$ Integer, where @(b) is the current content of box b, and statement {`@b =!` `v`} stores v into box b. An obvious semantic axiom is

$\{\}$
  {`@`b `=!` v}
$\{@(b) = v\}$

But, as we will see, a full semantic description of an abstract data type, even as simple as boxes, requires a little more than the obvious rules, and even the obvious rules can be tricky.

Example: stacks

Imagine a type of (last-in-first-out) stacks, with the following operations.

1. push(x,s) puts x onto the top of stack s (modifying s).

2. pop(x,s) removes the top item from stack s, and puts the result into variable x. (If s is empty, then it is an error to pop s.)

3. initialize(s) makes stack s empty. An empty stack contains no items.

4. empty(s) returns true if s is empty, and false if s is not empty.

An obvious property of the stack operations is

$\{\}$
  `push(`x`, `s`);`
  `pop(`y`, `s`);`
$\{y = x\}$

which says that, assuming no preconditions, if you push x onto stack s and then pop s, putting the result of the pop into y, then the values of y and x will be the same.

At any given time, the content of a stack of integers can be described as a list of integers, from the top of the stack to the bottom. For example, after pushing 2 and 9, in that order, onto an empty stack, the stack will contain list $[9, 2]$. That allows us to talk about the concept of the *current value* of a stack, and to compare two stack values. We will just write s in assertions to stand for the current value of stack s. An important property of stacks tells how push alters the stack.

empty(s) $\iff$ $s = [\,]$.

$\{A[s \leftarrow [\,]]\}$
  `initialize(`s`)`
$\{A\}$

$\{A[s \leftarrow x :: s]\}$
  `push(`x`,`s`)`
$\{A\}$

$\{$empty(s) & $A\}$
  `pop(`x`,`s`)`
$\{$false$\}$

$\{\neg$empty(s) & $A[s \leftarrow$ tail(s), $y \leftarrow x]\}$
  `pop(`y`,`s`)`
$\{A\}$

Figure 14.11: **Axioms for a stack data type in a Hoare style.** Postcondition *false* indicates an error, since it can never be reached. Notation $A[s \leftarrow \text{tail}(s),\ y \leftarrow x]$ indicates the result of replacing s by tail(s) and y by x in A.

$\{s = t\}$
  `push(`x`, `s`);`
$\{s = x :: t\}$

But it is not enough to say what the stack operations do; you also need to say what the stack operations do not do. For example, pushing x onto stack s does not change x, so you expect $\{x = 0\}$ `push(`x`,`s`)` $\{x = 0\}$ to be correct. That forces us to use fairly general rules, allowing any kind of assertion that is not affected by an operation to be in both the precondition and the postcondition. Say that $A[s \leftarrow e]$ is the result of replacing all occurrences of s in A by e. For example, if A is assertion $s = r$, then $A[s \leftarrow x :: z]$ is assertion $x :: z = r$. The action of push can be described by the following axiom.

$\{A[s \leftarrow x :: s]\}$
  `push(`x`, `s`);`
$\{A\}$

With enough properties such as this, you can say all of the properties that stacks must satisfy, and give a completely formal description of stacks. Figure 14.11 contains a set of axioms for stacks.

Even those axioms need to come with a warning. The value pushed onto a stack could be computed by an arbitrary expression. For example, there is nothing wrong with

```
push(f(y), s);
```

According to our axiom for Push,

$\{f(y) = 1$ and $f(y) :: s = r\}$
  `push(`$f(y)$`, `s`);`
$\{f(y) = 1$ and $s = r\}$

```
Type BinaryTree of <a> =
    leaf (leaflabel : <a>)
  | tree (treelabel    : <a>,
          leftSubtree  : BinaryTree of <a>,
          rightSubtree : BinaryTree of <a>)
%Type
```

Figure 14.12: **A parameterized type of binary trees, written in Cinnameg.** BinaryTree of T is a type of binary trees where each node is labeled by a value of type T.

You can see that the axiom is correct if f is a mathematical function. But what if function f has a side-effect, as functions are typically allowed to in imperative languages? Our axioms say nothing about the effects caused by f. Our semantic rules presume that functions and expressions have no side effects.

14.9. Polymorphic (parameterized) types

Figure 14.6 shows a definition of binary trees where each node is labeled by a string. But why should node labels be restricted to strings? What if you want, for example, a tree with an integer at each node, or with a pair holding an integer and a string? A more general tree would hold an arbitrary type of value at each node, whatever type you want to use. The binary tree type would be polymorphic.

Figure 14.12 shows a definition, in Cinnameg, of *parameterized* type BinaryTree, where BinaryTree of T is a type of binary trees where each node has a label of type T. When you parameterize a type, the function definitions usually change only to the extent that any type information written in them needs to be consistent with the new, parameterized, type. Functions treeSize and treeFlatten of Section 14.5 do not contain any type information in their definitions, and they will work for the parameterized BinaryTree tree type without any change.

14.9.1. Generics

Ideally, if you have decided to support polymorphism with type variables, then polymorphism should be built into the foundation of a language and its implementation. But often, the objective is to add polymorphism to an existing language that does not currently support it, making as few changes to the language and its implementation as are necessary. Generic polymorphism is a way of supporting polymorphism that requires a minimum of additional machinery. It dispenses with the need for type inference and unification, and with the need for the run-time implementation to be aware of polymorphism. As we will see, it has some good and some bad characteristics.

A first attempt: named types

Take a language, such as C, that does not support polymorphism. Suppose that you want to create a parameterized type Pair(S, T) in C, supporting some basic operations on ordered pairs. You can start with the module shown in Figure 14.13, which creates type

IntStringPair, a special case where S = int and T = string. Of course, that is not a polymorphic definition at all. When the need arises, you copy the entire definition and make some changes. For example, to create a pair type StringStringPair, holding two strings, just change the name of the type and change the definition of S. The required changes are simple and mechanical and it seems reasonable for a compiler to make them for you.

14.9.1.1. Generics in C++

C++ provides for polymorphism and parameterized types in its *template* facility. Figure 14.14 shows a definition of Pair in C++ in a more polymorphic way. To create a variable p that holds pair (2, "lucky"), you write the following.

```
Pair<int, string> p;
string str = "lucky";
p = buildPair(2, str);
```

A compiler can implement templates by recompiling the template definition for each new type that is used. For example, when it sees Pair<int, string> for the first time, it compiles the definition of Pair with S = int and T = string, choosing a new, internally generated, name for Pair, getLeft, etc. When it sees buildPair used, it compiles the definition of buildPair for the same types. It effectively converts a single parametric definition of buildPair into a collection of ad-hoc definitions, each of the same general form.

There are advantages and disadvantages of this approach to polymorhism. One advantage is that very little needs to be added to the language to provide for polymorphism. The C++ run-time system does not need to know about templates, although it normally knows the convention used to select names, so that it can show a type name as Pair<int, string> rather than the more cryptic name selected by the compiler. Type checking is performed when a definition is recompiled with specific types, so there is no real need for dealing with polymorphism in the type checker.

But there are disadvantages as well. One is that, in order to import a polymorphic definition, you need to import the source code for the definition, since it needs to be recompiled several times. If several modules use type Pair<int, string>, the definitions for that type can end up being recompiled for every one of those modules. If there are type errors, then they are reported at the places where the template is used, but occur within the template itself. So type errors can be difficult to understand. As an example, suppose that you make the following definition of the squaring function in C++.

```
template<typename T>
T sqr(T x)
{
  return x*x;
}
```

Then expression sqr(3) works, and produces 9, but expression sqr("abc") results in an error that is reported inside the template definition. To understand what it means, you need to read the template definition. Often, you use template definitions from libraries. Understanding those definitions and why they lead to errors can be difficult.

```
#define typeName   IntStringPair
#define getLeft    IntStringPair_getLeft
#define getRight   IntStringPair_getRight
#define buildPair  buildIntStringPair
typedef int S;    /* Left hand type */
typedef string T; /* Right hand type */
typedef struct typeName
{
  S left;
  T right;
} typeName;

S getLeft(typeName p)
{
  return p.left;
}

T getRight(typeName p)
{
  return p.right;
}

typeName buildPair(S lft, T rgt)
{
  typeName p;
  p.left = lft;
  p.right = rgt;
  return p;
}
```

Figure 14.13: **A definition of a pair type in C.** The type defined is IntStringPair, a pair containing an integer and a string. The #define lines use the C *preprocessor*, which performs substitutions. They tell the preprocessor to replace every occurrence of 'typeName' by 'IntStringPair', 'getLeft' by IntStringPair_getLeft, etc. The typedef lines create local types S, T and Pair, which are used in making the function definitions. The advantage of defining them is that, to create another type of ordered pairs StringString-Pair, all that is necessary is to change the definition of S to string, and to change the #define lines for typeName, getLeft, getRight and buildPair to the desired names. You can create several modules, one for each ordered pair type that you need, with minor (and mechanical) editing needed for each.

```
template <typename S, typename T>
struct Pair
{
  S left;
  T right;
};

template <typename S, typename T>
S getLeft(Pair<S,T> p)
{
  return p.left;
}

template <typename S, typename T>
T getRight(Pair<S,T> p)
{
  return p.right;
}

template <typename S, typename T>
Pair<S,T> buildPair(S lft, T rgt)
{
  Pair<S,T> p;
  p.left = lft;
  p.right = rgt;
  return p;
}
```

Figure 14.14: **A definition of a pair type in C++.** This turns the definition of Figure 14.13 into a polymorphic definition that is handled automatically by the compiler. The template lines create the type variables S and T. Pair$<S,T>$ is an ordered pair type similar to (S,T).

14.9.1.2. Generics in Ada

The Ada programming language uses a similar kind of generic definition, but requires the programmer to request recompiling a definition explicitly. Ada definition

```
generic
  type T is private;
function Sqr(X: in T) return T;

function Sqr(X: in T) return T is
  return X*X;
end Sqr;
```

defines a generic function Sqr with a single parameter (marked as an **in** parameter, indicating an input to the function). But before you can use Sqr, you need to request recompilation with a particular type. The following creates two versions, one that squares an integer and the other that squares a real number.

```
function SqrInt is new Sqr(Integer);
function SqrReal is new Sqr(Float);
```

A well-designed Ada program can avoid repeated compilation of a generic definition for the same type by requesting the desired versions in only one place.

Generics in Java

This section requires some knowledge of Java and object-oriented programming. Before generics were added to Java, you would create a general linked list class by making each thing in the linked list have type Object, meaning that it could be anything. Suppose you have created a linked list of called L that you know only contains integers, because that is all you have added to it. To get the head of L, you employ the head method.

```
Integer h = L.head();
```

But that does not work because L.head() has type Object, and cannot be stored in a variable of type Integer. To get the head, you would have to do an explicit conversion,

```
Integer h = (Integer) L.head();
```

requesting a run-time type check. If you have made a mistake, that mistake will only show up when it occurs while the program runs.

Generics allow the compiler to perform stronger compile-time type checking. Figure 14.15 defines a class of linked list cells and Figure 14.16 defines generic class LinkedList<T>. The machinery required to make the class generic appears to be minimal; you just attach a parameter <T> to the class and use T inside the class definition as the list element type. Also, type checking assures that, if L has type LinkedList<Integer> then expression `L.head()` has type Integer and `L.setHead("abc")` leads to a type error.

But the Java interpreter is unaware of generics. As far as the interpreter knows, there is no such thing as LinkedList<String>; there is only a class LinkedList, essentially the same as it would have been before generics were introduced. Consequently, you cannot make use of the parameter T in ways that would require the interpreter to know what it

```
class ListCell<T> {
  private ListCell<T> next;
  private T item;
  public ListCell(T anItem, ListCell<T> theNext) {
    item = anItem;
    next = theNext;
  }
  public T getItem() {
    return item;
  }
  public void setItem(T anItem) {
    item = anItem;
  }
  public ListCell<T> getNext() {
    return next;
  }
}
```

Figure 14.15: **Class ListCell<T>.**

```
class LinkedList<T> {
  private ListCell<T> firstCell;
  public LinkedList() {
    firstCell = null;
  }
  public LinkedList(T anItem, LinkedList<T> theNext) {
    firstCell = new ListCell<T>(anItem, theNext.firstCell);
  }
  private LinkedList(ListCell<T> cell) {
    firstCell = cell;
  }
  public boolean isEmpty() {
    return firstCell == null;
  }
  public T head() {
    return firstCell.getItem();
  }
  public LinkedList<T> tail() {
    return new LinkedList<T>(firstCell.getNext());
  }
  public void setHead(T x) {
    firstCell.setItem(x);
  }
}
```

Figure 14.16: **Class LinkedList<T>.**

is. Inside class LinkedList<T>, for example, you are not allowed to create a new object of class T or $T[\,]$ using **new**, because to do that, the interpreter would need to know T.

Suppose, for example, that you want to add another method to class LinkedList<T> that returns an array, of type $T[\,]$, holding the members of the linked list. That method needs to build the array, which it cannot do because it does not know T, so it cannot be written. (Java has a LinkedList class, not the same as the one defined here. It provides two toArray methods, one that returns an array of type Object[ ], thus avoiding the type problem, and the other taking an array of type $T[\,]$ as a parameter. The second method uses Java's reflection capabilities, similar to those described for Smalltalk in Chapter 21, to get the class of the parameter and create an instance of that class dynamically. Of course, neither is the toArray method that is really desired.)

Since the interpreter is unaware of the parameters, run-time type checks have to be relaxed. The compiler allows you to use class LinkedList directly (*with warnings*), and an object of class LinkedList<T> can be stored in a variable of type LinkedList. Consider the following lines.

```
LinkedList<String> A = new LinkedList<String>();
A = new LinkedList<String>("frog", A);
LinkedList B = A;
LinkedList<Integer> C = B;
C.setHead(3);
```

Variable B appears to have forgotten some information; it only remembers that it holds a linked list, not that it holds a linked list of strings. The conversion implicit in C = B really should fail at run time. But the interpreter has no way of checking it, and allows it to succeed. The effect of the last line, C.setHead(3), is that list A, nominally a list of strings, ends up holding an integer. That line does not throw an exception because the interpreter does not have enough information to allow it to know that something is wrong. Java's generic system is consequently type-unsafe, if you think it unreasonable that an integer can be put into a list of strings.

14.10. Exercises

14.1. Using the Distance type, write a function in Cinnameg that doubles a distance. That is, dbl(d) should produce a distance that is twice as long as distance d.

14.2. Would it make sense to have a function, of type (Distance, Real) → Distance, that adds a distance and a number? Why or why not? If so, what would the definition look like?

14.3. Would it make sense to have a function, of type (Real, Distance) → Distance, that multiplies a distance by a real number? Why or why not? If so, what would the definition look like?

14.4. Not every function that works on type Clothing needs to be written by cases. Write a definition of a function bigger such that bigger(a, b) is true if a has a larger weft than b, and is false otherwise. Figure 14.3 defines weft.

14.5. Write a Cinnameg function that produces the alphabetically smallest string in a nonempty binary search tree, using the type in Figure 14.8.

14.6. Using type BinarySearchTree from Figure 14.8, write two functions in Cinnameg.

(a) removeMin(t) should return a pair (x, t') where x is the smallest number stored in t and t' is a tree that holds all numbers in t except x. Presume that t is not empty. Make sure that t' is a binary search tree.

(b) remove(x, t) should return the tree that results from removing x from tree t. If x is not a member of tree t, then remove(x, t) should return t. (**Hint.** Start by thinking about trees with at least one empty subtree. Then deal with trees where both subtrees are nonempty. Use removeMin to help in that case.)

14.7. A *selection tree* is used to store a set of numbers so that the operation of finding the k-th smallest member of the set is efficient. A selection tree is the same as a binary search tree, but at each node an extra piece of information is stored telling the number of values in the subtree rooted at that node.

Define a type for selection trees. Define persistent versions of insert(x, t) and find(k, t), where insert(x, t) produces the result of inserting x into tree t, and find(k, t) returns the k-th smallest value in tree t. Use the sizes to determine where to find the k-th smallest. Be sure that your insert functions puts the correct size in a node.

14.8. Define a type Expression whose members represent expressions. Have a variant for a constant, holding a number, one for a variable, with a name, and variants for the sum and product of two given expressions, holding a pair of expressions. Now, using rules from Calculus, define a function derivative(e, x) that computes the derivative of e with respect to variable x, assuming that all other variables are independent of x. The result produced by derivative(e) should be another expression.

You will find that your results from derivative awkward. For example, they will include expressions such as $1 * x + 0$. Add another function that performs some simplifications on expressions, including $y + 0 =$ y and $y * 1 = y$. Are there other simplifications that you can add? Use your simplification function to simplify the result of the derivative function. (Note that derivative will be recursive. You do not want to simplify every time derivative produces an answer, or you will spend too much time simplifying. How can you arrange to simplify just once?)

14.9. A queue is a first-in-first-out list. A persistent implementation of queues has a constant emptyQueue (an empty queue), function isEmpty(q) (which returns true when q is empty), function insert(x, q) (which produces the queue that results when you add x to the end of queue q) and delete(q) (which produces ordered pair (x, q') consisting of the first member x of q and the queue q' that results when your remove x from q). The delete function should fail on an empty queue.

One way to represent a queue is via a pair of lists, (f, r). The contents of the queue, from front to back, consists of list f followed by the reversal of r. For example, pair ([1,2], [5,4,3]) represents a queue that contains [1,2,3,4,5], from front to back. To insert a value at the back, you just add it to the front of r. To remove a member when f is not empty, just remove the first member of f. If f is empty, then switch the representation from ([ ], r) to (reverse(r), [ ]).

Give a definition in Cinnameg of type Queue, representing a queue of strings, as well as definitions of emptyQueue, isEmpty, insert and delete, using this idea for representing queues.

14.10. What is the main difference in focus between the model-theoretic view of a type and the formal view of a type. Which view is more useful for imperative programming?

14.11. Write an axiomatic semantics for a type of object that behaves similarly to a box. Let's call it a store, and say that a store object holds one value, of type integer. Operation get(s, x) puts the current value of store s into variable x. Operation put(s, x) makes x the current value in store s. Design the semantics so that any implementation that satisfies the semantics must be a correct implementation of a store. You can use s in the semantics to indicate the current content of store s.

14.12. A first-in-first-out queue is a list that allows new things to be added at its back and values to removed from its front. Write axioms that define the semantics of a first-in-first-out queue, where the signature contains function empty(q) (true if q is empty) and procedures initialize(q) (make q empty), insert(x,q) (insert x in q, at its back) and remove(x,q) (remove the first member of q, and make x equal to it). Removal from an empty queue is an error.

14.13. Section 14.8 discusses an imperative implementation of a stack type. Suppose that you intend to use a persistent (in the sense of Section 4.9) implementation, where you cannot change anything. There is a constant emptyStack, a function push(x,s) that produces the stack that results from pushing x onto stack s, a function pop(s) that produces the result of popping stack s (or yields an error if s is empty), a function top(s) that returns the top of stack s (or an error if s is empty) and a predicate empty(s) that returns true just when s is empty.

Suppose that you already have a model-theoretic semantics of a stack as a list of values, read from top to bottom. Now you want a formal semantics of stacks that is consistent with the model-theoretic semantics. Write equational axioms that describe stacks, so that any implementation that satisfies the axioms will be faithful to the idea of a last-in-first-out stack.

14.14. Write a fully abstract implementation of binary search trees in a language, such as Java or C++, with which you are familiar. Include empty(t) (is t empty?) member(x, t) (is x in tree t?) and insert(x, t) (insert x into tree t). How can you ensure that your definition is fully abstract? Is anything defined behind the scenes by the compiler that prevents your definition from being fully abstract?

14.15. Enclose your definition of Queue from Problem 14.9 in a module, and make it fully abstract. Export emptyQueue, isEmpty, insert and delete, and nothing more.

14.16. Modify the definition of queues from Exercise 14.9 so that Queue is parameterized, and you can create a queue of any desired type of thing.

Chapter 15

Pure Functional Programming in Haskell

15.1. Introduction to Haskell

Haskell is a polymorphically typed, nonstrict, purely functional programming language whose development began in the late 1980s and continued through the 1990s and 2000s. It is undergoing constant improvement, but its core has remained quite stable. The following are some of its interesting features.

1. As it is nonstrict, it uses lazy evaluation as its normal evaluation policy.

2. Haskell uses the idea of layout to achieve a very compact syntax, and generally employs short and simple notation that makes programs easy to write. For example,

```
if x > y
  then n + 1
  else (m + 4)
         - 5*n
```

   uses indentation to show structure. The 'else' part has been broken into two lines for illustration. (Although you typically use layout, you can also use explicit braces, $\{a; b; c\}$ to group things, separating them by semicolons. In fact, the initial stage of the Haskell compiler just inserts braces and semicolons based on the layout.)

3. Haskell has a polymorphic type system, providing support for data abstractions and type variables, with type inference, as studied in Chapter 13.

4. Because Haskell is purely functional, it needs to find ways to do such usually imperative features as input and output without the need for side effects in the language. How that is accomplished involves some clever trickery that demonstrates the power that can lie in functional programming.

This chapter gives a brief introduction to Haskell. You will find much of Haskell to be minor syntactic variations on what we have covered earlier.

$x + y$, $x - y$, $x * y$, /	arithmetic binary operators
x 'div' y, x 'mod' y	integer quotient and remainder operators
$x\hat{}y$, $x * *y$	power operators (^ raises to integer power, ** to real power)
$x > y$, $x < y$, $x >= y$, $x <= y$	comparisons
$x == y$, $x/ = y$	equality and inequality tests
$x\&\&y$, $x\|\|y$, not x	Boolean **and**, **or** and negation
$h : t$	$h : t$ is the list whose head is h and whose tail is t.
$x + +y$	list concatenation
$x!!n$	$x!!$ n is the n-th member of list x, counting from 0. For example, $x!!0$ is the head of x.
head x, tail x	the head or tail of list
length x	the length of a list
fst p	the first member of an ordered pair. (Unlike Cinnameg, Haskell does not consider an ordered triple to be a special kind of ordered pair, so you cannot use fst to give the first member of a triple.)
snd p	the second member of an ordered pair
$f.g$	$f.g$ is the composition of functions f and g. That is, $(f.g)(x) = f(g(x))$.

Figure 15.1: **Some standard functions and operators.** The standard functions are typically curried. Expression x 'op' y abbreviates (op x y), and $x + y$ abbreviates $(+)(x, y)$. Haskell is case-sensitive, and these function names need to be written exactly as shown.

15.2. Lexical issues

Identifiers begin with a letter and can contain letters, digits and special characters _ and '. Symbolic operator names are one or more of the special symbols in string "`!#$%*+-./<>=?\^|:~`". Comments can either begin with `--` and extend to the end of a line or be enclosed in {- and -}.[1]

15.3. Expressions and values

Haskell supports typical values such as integers (32), real numbers (32.5) and characters ('a'). Boolean constants are True and False. Be sure to capitalize them. Haskell supports ordered pairs such as (5, True) and list such as $[1, 2, 3]$ in the same form as Cinnameg. A string is a list of characters; "red" abbreviates ['r', 'e', 'd'].

Each program starts by importing the it standard prelude, and Figure 15.1 shows a few functions and operators that are defined there. Juxtaposition indicates function application, and associates to the left, just as in Cinnameg. Expression (\x -> E) is the function that takes parameter x and produces the value of expression E. For example, the value of (\y -> y + 1) is a function that returns one larger than its parameter.

[1] There is a *literate* style of Haskell in which all lines are presumed to be comments except those that begin with >. We will use the ordinary style.

Expression `if` *a* `then` *b* `else` *c* is *b* if *a* is True and is *c* if *a* is False. Expression `let` *x* `=` *A* `in` *B* yields the value of *B*, where *x* has value *A*. For example,

```
let amphibian = "frog" in
  amphibian ++ "s and more " ++ amphibian ++ "s"
```

defines amphibian to be "frogs and more frogs". Since Haskell is always lazy, the value of a variable is only computed when it is used. List [1 . . 5] is the list $[1, 2, 3, 4, 5]$. List comprehensions (Section 9.10.4) allow any number of generator parts (x <- ys) and filter parts (boolean expressions), separated by commas. For example,

- `[2*n | n <- [2..5]]` yields $[4, 6, 8, 10]$,
- `[x | x <- [1..8], isEven x]` yields [2,4,6,8] and
- `[(x,y) | x <- [2..4], y <- [2..4], x < y]` yields [(2,3), (2,4), (3,4)].

15.3.1. Patterns

The left-hand side of an equation can be a *pattern*. For example,

(x,y) = p

makes x the first member of ordered pair p and y the second member. You can use pairs and lists, or expressions of the form *h*:*t*, or constants. You can also use patterns such as $n + 1$. An underscore is a "don't care" variable. Each variable can occur only once, so no equality checks are performed.

A case-expression tries several pattern matches against one value, yielding the value associated with the first pattern that matches. Expression

```
case x of
  []  -> 1
  _:_ -> 2
```

produces value 1 when x is an empty list and 2 when x is a nonempty list.

15.4. Definitions

A definition has the form of an equation where the left-hand side is a name or a function name applied to one or more variables or patterns. Looking ahead toward types, a stand-alone definition (not part of a larger definition) is typically (but not necessarily) preceded by an indication of its type. You can include more than one equation by writing them one after another; they are tried in the order written. Here are some examples.

```
num :: Int -- num has type Int
num = 43 + 19

double :: Int -> Int
double n = 2*num

reverse :: [Int] -> [Int]
reverse []     = []
reverse (x:xs) = reverse xs ++ [x]

max :: Int -> Int -> Int
max x y = if x > y then x else y
```

To use provisos for equations (called *guards* in Haskell), write the left-hand side once, but precede each right-hand side by a bar, the condition under which it works, and an equal sign. For example, an alternative definition of max is as follows.

```
max x y
  | x > y     = x
  | otherwise = y
```

A case labeled **otherwise** is always selected. If none of the cases have true guards, the next equation or group of equations is tried. After an equation or list of guarded right-hand sides you can include a **where** clause that defines variables that can be used throughout that equation. For example, maxabs(x,y) = max($|x|$, $|y|$) can be defined as follows.

```
maxabs x y
  | ax > ay   = ax
  | otherwise = ay
    where
      (ax, ay) = (absval x, absval y)
      absval z
        | z >= 0    = z
        | otherwise = -z
```

Notice that the definitions of ax and ay use absval before it is defined. Haskell allows definitions to be written in any order and used before they are defined. Of course, definition

```
maxabs x y = max(abs x, abs y)
```

using the standard abs function is much simpler.

15.5. Types

Haskell is a strongly statically typed language. here are covered in later chapters of this book. Types have names that begin with an upper case letter, and include Int, Float (real numbers), Bool and Char. Ordered pair (2,True) has type (Int, Bool). List [1,2,3] has type [Int] and "jump" has type [Char]. You can give a new name to an existing type using a **type** definition. For example, the standard prelude includes

```
type String = [Char]
```

which defines String to be another name for [Char].

Create a new type using a **data** definition. The left-hand side is the name of the new type and the right-hand side has a constructor name followed by zero or more types, where *the type and constructor names are required to start with upper case letters.* For example,

```
data Name = MakeName String
```

defines type Name to be represented as a string. Constructor MakeName has type String → Name. To get the string that a given Name represents, use pattern matching. For example,

```
plural(MakeName str) = MakeName(str ++ "s")
```

makes one Name from another by adding an 's'. A type can have variants, as discussed in Section 14.4. A simple example is an enumerated type, such as

```
data Flower = Orchid | Rose | Gladiolus
```

Types can be recursive. For example, the following defines a binary tree type with nodes labeled by integers.

```
data BinaryTree = EmptyTree |
                  Node Int BinaryTree BinaryTree
```

The second variant says that a binary tree constructed by Node contains an integer and two other binary trees; expression (Node x L R) builds a tree with root label x, left subtree L and right subtree R. The number of nodes in a tree is easy to compute using primitive recursion.

```
numNodes :: BinaryTree -> Int
numNodes EmptyTree      = 0
numNodes (Node _ t1 t2) = numNodes t1 + numNodes t2
```

15.6. Polymorphism

Haskell is polymorphically typed; a type variable has a name that is an identifier. For example, the head function has polymorphic type [a] -> a. Types can be parameterized as discussed in Section 14.9. Here is a parameterized version of the binary tree type; type Tree t is a tree whose nodes are labeled by values of type t.

```
data Tree t = EmptyTree |
              Node t (Tree t) (Tree t)
```

Now to sum the numbers in the tree:

```
sumTree :: Tree Int -> Int
sumTree EmptyTree      = 0
sumTree (Node x t1 t2) = x + sumTree t1 + sumTree t2
```

A more interesting function numbers the nodes of a tree in preorder, replacing each node label x by a pair (n, x), where n is the number that this node would have in a preorder traversal of the tree, starting the numbering at 1. Function numberTreeHelp takes an extra parameter that is the label to attach to the root, and it returns pair (m, r) where r is the labeled tree and m is one larger than the last label that was used in r (or is just n if r is an empty tree).

```
numberTree :: Tree a -> Tree (Int, a)
numberTree t = snd(numberTreeHelp 1 t)

numberTreeHelp :: Int -> Tree a -> (Int, Tree(Int, a))
numberTreeHelp n EmptyTree = (n, EmptyTree)
numberTreeHelp n (Node x t1 t2)
  = (m2, Node (n,x) nt1 nt2)
    where
      (m1, nt1) = numberTreeHelp (n+1) t1
      (m2, nt2) = numberTreeHelp m1 t2
```

Haskell uses the word *class* for a concept similar to a genus in Cinnameg. Think of a class as a set of types. For example, standard Haskell class Eq contains those types that support equality (==) and inequality (/=) tests. Standard class Ord contains types on which you can perform order tests (<, >, etc.).

To require a type variable to be a member of a given class, write *conditions* `=>` *type*, where each condition has the form $C\ v$, indicating that variable v must belong to class C. If there two or more conditions they are required to be in parentheses and separated by commas. For example, the Quicksort algorithm is as follows.

```
sort :: Ord a => [a] -> [a]
sort [] = []
sort x:xs = sort small ++ [x] ++ sort large
  where
    small = [y | y <- xs, y <= x]
    large = [y | y <- xs, y > x]
```

Figure 15.2 illustrates how to create a new class, indicating the operations that all types in the class share. To add a type to a class, you say how to perform the operations that are part of the class signature. Any functions that you do not define take their default definitions from the class definition. For example,

```
instance Eq Bool where
  -- Definition of ==:
  True == True   = True
  False == False = True
  _ == _         = False
```

says that type Bool belongs to class Eq, and defines the == operator for it, accepting the default definition for the /= operator.

15.7. Modules

A module has the form

```
module name (exports) where
  ...
```

where *exports* tells the names that this module exports and the body (. . .) gives definitions. The things that can be exported are as follows.

1. A constant, function or type name exports that constant, function or type.
2. Form T`(..)`, where T is a type, indicates that T is exported along with its constructors (including the ability to use the constructors in patterns). Without `(..)` only type T is exported, without its constructors.
3. Form `module` M indicates that all exported bindings in module M are exported to any module that imports the current module. If M is the name of the current module, then all bindings in the current module are exported.

```
class Eq t where     -- Definition of class Eq.
                     -- If type t belongs to class Eq,
                     -- then the following operations
                     -- are defined for type t.
  (==), (/=) :: t -> t -> Bool
  -- The following are default definitions
  x == y  = not(x /= y)
  x /= y  = not(x == y)

class (Eq t) => Ord t where    -- Definition of Ord
                               -- All types in Ord also
                               -- belong to Eq.
  compare              :: t -> t -> Ordering
  (<), (<=), (>), (>=) :: t -> t -> Bool
  min, max             :: t -> t -> t
  -- The following are default definitions
  x < y  = compare x y == LT
  x > y  = compare x y == GT
  x <= y = compare x y /= GT
  x >= y = compare x y /= LT
  max x y
    | x > y     = x
    | otherwise = y
  min x y =
    | x > y     = y
    | otherwise = x
```

Figure 15.2: **Definitions of classes Eq and Ord.** The definition gives the class name and a variable that represents an arbitrary type in the new class. The variable is used in the polymorphic types of functions (and possibly other kinds of things) that make up the signature of the class. You can include default definitions as well. Definitions made elsewhere will override them.

Ordered types must also possess equality tests, so class Ord is a subclass of Eq. Adding a type to Ord automatically adds it to Eq as well. Type Ordering is defined by

```
data Ordering = LT | EQ | GT
```

```
module Set (Set, set, insert, member) where
  type Set = Tree
  data (Ord a) => Tree a = EmptyTree | Node a (Tree a) (Tree a)
  set :: [a] -> Tree a
  set []    = EmptyTree
  set (h:t) = insert h (set t)
  member :: a -> Tree a -> Bool
  member x EmptyTree = False
  member x (Node k lft rgt)
    | x < k = member x lft
    | x > k = member x rgt
    | otherwise = True
  insert :: a -> Tree a -> Tree a
  insert x EmptyTree = Node x EmptyTree EmptyTree
  insert x (Node k lft rgt)
    | x < k = Node k (insert x lft) rgt
    | x > k = Node k lft (insert x rgt)
    | otherwise = Node k lft rgt
```

Figure 15.3:

Figure 15.3 shows a sample module that provides sets, represented as binary search trees. To import the Set module, use

```
import Set
```

You can import only selected things or all but selected things. For example,

```
import Kangaroo(Kangaroo(..), hop)
import Wallaby hiding (jump, goOnAllFives)
```

imports only type Kangaroo (with constructors) and hop from the Kangaroo module, and imports everything except jump and goOnAllFives from Wallaby.

Haskell uses names that include the module. When you create module Set, for example, what you really get are Set.Set, Set.insert and Set.member. Typically, you prefer to use short names Set, insert and member, and that is what you normally get. If you want to use the long names, then import the module *qualified*, as follows.

```
import qualified Set
```

Now you will need to write Set.insert to use the insert function from the Set module.

15.8. Actions and monads

Imperative programming is based on actions. An action can change a variable, show something on the console, modify a file, etc. But a functional program only computes values, and

pure functional programming languages cannot allow a program to request that an action be done. For example, you cannot ask to print something or to create a new file, since those things do not just involve computing a value, or giving a name to a value. That is a serious problem, since a general purpose programming language should allow a program to interact with its environment in ways that involve making changes to the environment.

One solution is to back away from pure functional programming and to add actions to the language, yielding a language that is only partially functional. That approach is taken by Scheme and SML, for example. But there are some unpleasant consequences of that. A functional language is intended to offer mathematical simplicity. For example, if you know that two expressions A and B are equal then you should be able to replace A by B or B by A without changing the results that a program produces. But that rule does not work in an imperative language. When actions are introduced, proofs that programs are correct, as discussed in Chapter 24, have to take imperative aspects into account; the reasoning is usually sound only if no actions are involved. Mathematical simplicity is lost.

15.8.1. Delayed actions

But it turns out that actions can be added to a programming language in such a way that the language remains purely functional. The key idea is to provide actions, but not to provide a way to perform an action. You can build an action, but you cannot run it! There is nothing imperative about a function that, for example, takes an integer n and produces an action that would print n, *if that action were run*. All you are doing is producing values of a type such as Action. Because the actions do not do anything until they are triggered later, we will call them *delayed actions*.

Of course, *somebody* needs to be able to run a delayed action, or there is no point to it. A typical approach is to provide a user interface to a functional language where you can evaluate expressions. If the result of evaluating an expression is a delayed action, then the user interface (not the program) runs the action. The action produced by a function can contain requests for other actions produced by other functions. The user interface calls those other functions to ask what to do.

An imperative program interacts with the operating by making system calls; the calls are initiated by the program and return answers from the operating system. Delayed actions reverse that. The user interface, acting as a proxy for the operating system, calls the functional program to ask what to do. On receiving an answer (a delayed action), it performs the requested action and asks what to do next. An action can have a result. For example, reading a file yields some data. Action results are passed as parameters back to the functional program, where it can decide what to do next based on the what it is told about the previous action's result.

15.8.2. Valueless actions

We start by looking at actions that do not produce answers. Haskell function putStrLn takes a string and produces a valueless delayed action that, when run (by the user interface), will write that string, followed by a newline. So if printCompliment is defined by

```
printCompliment adj = putStrLn "You are " ++ adj
```

then (printCompliment "wise") is a delayed action that, when run, will write "You are wise".

One of the fundamental concepts of imperative programming is performing actions in sequence; first do action A, then action B. You can sequence delayed actions in Haskell using the **do** construct. If A, B and C are valueless delayed actions, then expression

```
do A
   B
   C
```

produces, as its value, a delayed action that performs A, B and C, in that order. You can sequence any number of actions. For example,

```
do printCompliment "wise"
   printCompliment "generous"
```

creates a valueless delayed action that writes two compliments. It is important to keep in mind that the do construct does not actually do anything. It is an expression whose value is a delayed action that will do things, when the action is performed.

15.8.3. Tests and loops

You can simulate simple tests and loops by deciding how to build a delayed action. For example, suppose that you want to print "You are wise" n times. You will want a *do-nothing* action, to make it possible for n to be 0. It can be defined as follows in Haskell.

```
skip = return ()
```

Then to write the compliment n times, it suffices to chain together n delayed actions.

```
printWiseNTimes n =
  if n == 0
    then skip
    else do printCompliment "wise"
            printWiseNTimes (n-1)
```

Notice the need for another do in the else part to sequence delayed actions. Think of it as similar to a compound statement. Once again, printWiseNTimes(3) does not actually write anything. It is an expression whose value is a delayed action that, when run (if ever) will write n lines.

15.8.4. Actions that produce answers

Sometimes an action yields an answer. If a is an action that yields a result of type T, then a has type IO T. Delayed action

```
return v
```

does nothing but yield answer v. (It does not return from a function! It just yields a delayed action.) Haskell provides delayed action getLine that, when run, will read a line of text and return it (sans end-of-line character). To get the value that an action yields within a **do** expression, use form x `<-` action. For example, readTwoLines, defined by

```
readTwoLines =
  do line1 <- getLine
     line2 <- getLine
     return line1 ++ line2
```

is a delayed action that, when run, reads two lines and returns the two of them concatenated together. If you want to run an action and get its answer, you will need to use the <- notation and give a name to the answer. It would not work to do

```
readTwoLines =
  return getLine ++ getLine
```

Remember that getLine is a delayed action. Expression getLine ++ getLine asks to concatenate two delayed actions, not two strings.

15.8.5. Cautions

Delayed actions and the do construct make it easy to imagine that you are doing imperative programming. For example, you can define your own while-loop construct as follows, where both the condition to test and the body are given by delayed actions. (If the condition were just an expression of type Boolean, it would always have the same result.)

```
while cond body =
  do t <- cond
     if t
        then do body
                while cond body
        else skip
```

Now suppose that you want to read lines and only write those that begin with '#'. Using while, you can do that as follows, using Haskell function isEOF to test for end-of-file.

```
printSharpLines =
  while (do t <- isEOF
            return(not t))
        (do line <- getLine
            if line /= [] && head line == '#'
              then putStrLn line
              else skip)
```

But working with delayed actions in a functional language is really quite different from working in an imperative programming language. Haskell does not allow you to change the value of a variable, once it is given a value. (Allowing that would take Haskell out of the realm of functional programming.) Suppose that you decide to add up the values in a list of numbers as follows.

```
sum lst =
  do x <- lst
     s <- return 0
     while (return (x /= []))
           (do s <- return (head x + s)
               x <- return (tail x))
     return s
```

The loop tries to change s and x. But what happens is that new variables s and x are created in the loop body that shadow those outside the loop. The test keeps looking at the original list. So, if the original list is not empty, the loop goes forever. You are better off using recursion, and remembering never to try to change an existing variable value.

15.9. Monads and notions of sequencing

The **do** construct and notation `x <- action` are not primitive; they are defined in terms of more fundamental ideas. Haskell operator `>>=` performs sequencing. When used to sequence actions, its type is

```
(>>=) :: IO s -> (s -> IO t) -> IO t
```

The idea is that a `>>=` f produces an action that, when run, will (1) perform action a, yielding result r; (2) compute $b = f(r)$, where b is another action; (3) perform action b; and (4) produce the result of action b as its own result. Construct

```
do x <- a
   b
```

is converted to (a `>>=` (`\`x `->` b)). Construct

```
do a
   b
```

just ignores the result produced by a. It is converted to (a `>>=` (`\_ ->` b))

But there is more than one notion of sequencing, and more than one kind of thing that can be sequenced. You can string together pearls on necklace, floats in a parade, or steps in an assembly process. A *monad* represents a particular definition of sequencing, and it is defined in terms of how operator `>>=` and two related functions, return and fail, work. We have already seen how Haskell deals with actions. We look at two other examples, handling errors and backtracking, to show what monads offer.

15.9.1. Handling errors

Haskell does not support exception handling. But it does provide type Maybe t, whose values can be of the form Just(v) (where v has type t) or Nothing (where Nothing indicates a lack of an answer). In addition to the return function, which just produces an answer, a monad offers function fail(s), indicating some kind of failure, with string s indicating the reason for the failure. Like (`>>=`) and return, the definition of fail depends on the monad.

Imagine that you intend to compute expressions A and B, where computation of B depends on the value produced by A. If A produces Nothing, you probably do not want to continue with B; just produce Nothing for the entire computation, without computing B at all. To that end, define

```
(Just x) >>= f = f x
Nothing  >>= f = Nothing
return x       = Just x
fail s         = Nothing
```

Then, using the definition of **do** expressions,

```
do x <- Just 1
   return x + 1
```

yields Just(2), but

```
do x <- fail "no result"
   return x + 1
```

yields Nothing.

15.9.2. Backtracking

Chapter 16 introduces backtracking and the Cinnameg Backtrack construct that employs it. Backtracking allows an expression to produce more than one value by forking into several branches and producing a different value in each branch. But it is not necessary to have direct support for backtracking; you can simulate backtracking by letting each expression produce a list of values. For example, an expression that produces list $[1, 2, 3]$ can be thought of as producing three different values, one in each of three branches. A key issue is how to sequence expressions that simulate backtracking. If expression A produces $[1, 2]$ and B produces $[5, 9]$, then $x = A$ followed by $y = B$ should be able to produce (x, y) values [(1,5), (1,9), (2,5), (2,9)]. The backtrack monad implements sequencing that way. For example, expression

```
do x <- [1,2]
   y <- [5,9]
   return x+y
```

yields list $[6, 10, 7, 11]$. Definitions for the backtracking monad are as follows.

```
xs >>= f = concat (map f xs)
return x = [x]
fail s   = []
```

Standard Haskell function concat takes a list of lists and produces their concatenation, so concat $[[1, 2], [3, 4], [5]] = [1, 2, 3, 4, 5]$. Expression

```
do x <- [1,2,3]
   return (x + 1)
```

is translated to ([1, 2, 3] >>= (\x -> return(x+1))), which is computed as follows.

$$
\begin{aligned}
&[1, 2, 3] \texttt{ >>= } (\backslash x \texttt{ -> } \text{return}(x + 1)) \\
&\quad = \text{concat}(\text{map } (\backslash x \texttt{ -> } [x + 1])\ [1, 2, 3]) \\
&\quad = \text{concat}([[2], [3], [4]]) \\
&\quad = [2, 3, 4]
\end{aligned}
$$

15.9.3. The Monad class

Class Monad t contains types such as IO t, Maybe t and $[t]$ (for backtracking) that are defined to be monads. For example, the definition of the backtracking monad is as follows, where [ ] indicates the list-type constructor.

```
instance Monad [] where
  xs >>= f = concat (map f xs)
  return x = [x]
  fails s  = []
```

(Notice that Monad and [] are both parameterized. Saying that constructor [] belongs to Monad implicitly says that $[t]$ belongs to Monad t for every type t.) To create a new monad, it is just a matter of adding a new type to Monad.

15.10. Exercises

15.1. Write a definition in Haskell of each of the following functions.

(a) The factorial function.

(b) The Haskell ++ operator. You can use binary operator notation in definitions, so define what x ++ y is. To test this, you will need to hide the standard ++ operator. Write

```
import Prelude hiding (++)
```

(c) Function dup n x, which produces a list of n copies of x.

(d) removeLeading r xs, which produces the list obtained from list xs by removing all initial occurrences of r. (For example, removeLeading 'z' "zzzazn" = "azn".)

(e) The map function, map f [a,b,c] = [$f\,a$, $f\,b$, $f\,c$].

(f) Function prefix x y, which yields True just when list x is a prefix of list y.

15.2. Some imperative aspects can be simulated by higher-order functions. Standard Haskell function `until` p f x computes the sequence of values $y_0 = x$, $y_1 = f(y_0)$, $y_2 = f(y_1)$, etc. until it hits the first value y_i such that y_i is true, and returns that value. It is similar to a while loop where predicate p tells when to stop the loop and function f tells how to update.

(a) Write a definition of until.

(b) Write a definition of reverse using until to perform the looping. Use the algorithm that shunts values from one list to another.

(c) Write a definition of factorial using until to do looping.

15.3. Suppose that `search` x s is intended to yield True if string x occurs as a contiguous substring of string s, and False otherwise.

(a) Write a definition of search, with a type.

(b) Suppose that you declare search to have polymorphic type [a] `->` [a] `->` Bool. Does the definition of search that you wrote for part (a) work for this polymorphic type? Why or why not?

15.4. A directed graph on n vertices can be represented as a list of the form $[L_1, L_2, \ldots, L_n]$, where, for $i = 1, \ldots, n$, L_i is a list of all vertices j so that there is an edge from i to j.

Write a Haskell definition of a function reachable u v g, which returns true if there is a directed path from u to v in g (where g is given as a list of lists, as explained).

15.5. It is important to be able to convert algorithms that are described in an imperative way into a functional style. This exercise has you do that. First, the following is an imperative description of an algorithm.

It is easy to modify the representation of a directed graph to include weights on the edges (indicating distances to be traveled). Just let each list member be a pair (w, j) telling the weight of the edge to j. For example, [[(30,2), (25,3)], [(110,1)], [ ]] describes a graph with three weighted edges: from 1 to 2 (weight 30), from 1 to 3 (weight 25) and from 2 to 1 (weight 110).

There is a well-known algorithm, Dijkstra's algorithm, to find the length of the shortest directed path between two given vertices. It performs a simulation. Imagine sending a signal along an edge. If the edge has weight w, then it takes w seconds to travel from one end to the other of that edge. To find the distance from u to v, start the simulation by sending a signal on every edge that comes out of u.

An *event* is the arrival of a signal at a vertex. The idea is to keep a list of events, each holding the time (from the beginning of the simulation) at which it will occur and the vertex where the signal will arrive. Initially, the list of events should be [(0,u)], forcing a signal to arrive at the start vertex u at time 0. Now process the events in chronological order. To process event (t, x), where t is a time and x is a vertex number, do the following.

(a) If $x = v$ (the end vertex) then the simulation is done. The distance from u to v is t.

(b) If x has not previously been reached by a signal and $x \neq v$, then send a signal along each edge coming out of x by scheduling more events, indicating the arrivals of those signals at the other ends of the edges. If there is an edge from x to y of weight w, then add event $(t + w, y)$ to the event list.

(c) Record that a signal has reached vertex x.

Write the shortest distance function in Haskell. A simple way to simulate a variable that a function wants to change is to pass that variable to the function (giving the variable's "current value") and to return an updated value from the function. A function can return several values by returning a tuple. You will probably want some helper functions. Give a clear description of what each one is intended to do. Do not make someone reverse engineer your definitions to understand what they are doing.

15.6. Write a Haskell module that provides type Queue T, a first-in-first-out queue whose members have type T. Provide functions to insert a value, remove the first value, look at the first value and ask whether the queue is empty. How can you insert or remove a value in a purely functional language where no data structures can be changed?

15.7. A formula of propositional logic is *valid* if it is true for all values that its variables could take on. For example, $P \Rightarrow P$ and $P \Rightarrow (Q \Rightarrow P)$ are both valid. This exercise is to write a definition, in Haskell, of function valid(r) that returns true if r is a valid propositional formula.

(a) A *formula* can be a variable name (a single letter), or have the form $\neg A$ or $A \vee B$ or $A \wedge B$ or $A \Rightarrow B$, where A and B are formulas. Write a Haskell

$$(\wedge L): \quad \frac{(A,\, B,\, \alpha \vdash \beta)}{(A \wedge B,\, \alpha \vdash \beta)} \qquad (\wedge R): \quad \frac{(\alpha \vdash A,\, \beta),\ (\alpha \vdash B, \beta)}{(\alpha \vdash A \wedge B,\, \beta)}$$

$$(\vee L): \quad \frac{(A,\, \alpha \vdash \beta),\ (B,\, \alpha \vdash \beta)}{(A \vee B,\, \alpha \vdash \beta)} \qquad (\vee R): \quad \frac{(\alpha \vdash A,\, B,\, \beta)}{(\alpha \vdash A \vee B,\, \beta)}$$

$$(\neg L): \quad \frac{(\alpha \vdash A,\, \beta)}{(\neg A,\, \alpha \vdash \beta)} \qquad (\neg R): \quad \frac{(A,\, \alpha \vdash \beta)}{(\alpha \vdash \neg A,\, \beta)}$$

$$(\Rightarrow L): \quad \frac{(\alpha \vdash A,\, \beta),\ (B,\, \alpha \vdash \beta)}{(A \Rightarrow B,\, \alpha \vdash \beta)} \qquad (\Rightarrow R): \quad \frac{(A,\, \alpha \vdash B, \beta)}{\alpha \vdash A \Rightarrow B,\, \beta}$$

Figure 15.4: **Rules of inference for propositional logic.** These rules work with sequents, allowing you to prove that the sequent below the line is valid, assuming you have already proved that all of the sequents above the line are valid. Greek letters α and β stand for arbitrary lists of formulas. You are free to reorder the lists so that the formula that you are interested in is first.

There are two rules for each operator ($\wedge$, $\vee$, $\neg$ and $\Rightarrow$). One is used to introduce that operator into a formula on the left-hand side of a sequent, and the other introduces the operator into a formula on the right-hand side. The names of the rules express what they introduce. For example, rule ($\wedge L$) adds a new $\wedge$ operator into a formula on the left-hand side of a sequent.

definition of type Formula. Provide an equality testing function for formulas. Two formulas are equal if they are identical in form. Provide a function that takes a string and produces a Formula

(b) A *sequent* has the form $L \vdash R$ where L and R are lists of formulas. It means, if all of the formulas in list L are true, then at least one of the formulas in list R is true. Either side can be an empty list. (An empty list on the left means that there are no requirements. An empty list on the right guarantees a false sequent, since it is surely not the case that one of no things is true.) The order of formulas in the lists does not matter; you are free to reorder them, and all that matters is whether it is on the left-hand side or the right-hand side.

A sequent is valid if it is true for all values of the variables. Define type Sequent. (Ignore symbol $\vdash$. A sequent is just given by a pair of lists of formulas.)

(c) A sequent is *basic* if there is a formula that occurs in both the left list and the right list. For example, sequent $(P \vee Q, P \vdash P \vee Q)$ is basic since $P \vee Q$ occurs on both sides. Every basic sequent is valid. (Why?) Define a function basic(s) that yields True if s is a basic sequent, and False if not.

(d) Table 15.4 shows rules of inference that allow you to prove sequents are valid, assuming some others are already known to be valid.

To prove that a sequent is valid, keep a goal list of sequents that need to be proved. To prove that formula A is valid, start the goal list holding just one sequent, $(\vdash A)$, where the left-hand side is empty.

Remove a sequent from the goal list and examine it. If the sequent is basic, you are finished with it; move on the the remaining goals. Otherwise, if the

sequent has no formula on either side that has an operator (so all formulas are just variables) then it is not valid. Conclude that your original sequent is not valid.

If the sequent is not basic, but it has at least one formula that is not just a variable, select a formula, either on the left or right side, that has an operator ($\neg$, $\vee$, $\wedge$ or $\Rightarrow$). There is an inference rule that allows you to introduce that formula. Determine what other sequents would need to be proved valid in order to use that inference rule. Add those sequents to your goal list. (There will be one or two new sequents.) Continue, selecting a sequent to prove.

For example, to prove that ($P \vdash Q \Rightarrow P$) is valid, select formula $Q \Rightarrow P$ from the right-hand side. Rule ($\Rightarrow$ R) says to add new sequent ($Q, P \vdash P$) to the goal list; if you prove that sequent, you can conclude ($P \vdash Q \Rightarrow P$). Since ($Q, P \vdash P$) is basic, you are done.

15.8. What is the advantage of using monads over simply adding imperative constucts to a functional language?

15.9. Convert the definitions of each of the following to its underlying form using `>>=`.

(a) printWiseNTimes (page 216)

(b) readTwoLines (page 217)

15.10. Haskell allows an arbitrary pattern on the left-hand side of `<-`. As a first approximation,

```
do p <- a
   b
```

is translated to (a `>>=` (`\`p `->` b)). But there is a problem with that. Although Haskell does allow a pattern after `\`, such patterns are restricted to those that cannot fail to match. It is not allowed to have a function (`\x:xs ->` ...) since pattern x:xs can only match a nonempty list. To deal with that, a failed pattern match is defined to yield fail(s) where s is a string that says the match failed. Show how to translate the above do expression, for an arbitrary pattern p, so that it has the desired effect.

15.11. Write a Haskell definition of function conditional(A, B, C) that simulates a conditional (if-then-else) statement where the condition A to be tested is a delayed action that produces a boolean result, and the two options B and C are valueless delayed actions.

15.12. Would the while function make any sense for the maybe monad or the backtracking monad, or is there something about the concept of a loop that says it should only be used to sequence actions?

15.13. The maybe monad acts as a stand-in for exception handling. Can you see any reason for using it instead of providing direct support for exception handling in a purely functional language?

15.11. Bibliographic notes

Thompson [95] and Bird [15] are introductory texts in functional programming using Haskell. Hudak, Hughs, Jones and Wadler [53] give a history of Haskell. You can find information on Haskell, including information on getting implementations, tutorials, and the Revised Haskell 98 Report at http://www.haskell.org/.

Chapter 16

Dealing with Failure

16.1. Failure

What should a program do when it attempts to divide by 0? An obvious (but naive) answer is that the program should be stopped immediately, since it has clearly done something wrong. But that is a heavy-handed way of dealing with the problem. The person using the program might not be the one who wrote it, and might have no interest in hearing about what caused the error. It is true that something is wrong, but the program should really be given some opportunity to recover from the error. Software that suddenly stops running without warning tends to be unpopular.

One approach to allowing recovery is to design each subprogram or operation that might encounter trouble so that it returns a special value, or sets a flag, indicating an error. If IEEE floating point arithmetic is used, division by 0 yields a special value NaN (*Not a Number*), and the program can test the outcome of an operation to see whether the result is NaN. Status results tend to be pervasive and a carefully written program using status values must make frequent checks to see that no errors have occurred. A significant portion of the program has to be devoted to checking and dealing with status values; programs can be cluttered with this extra code that is rarely executed, and whose existence is really peripheral to the logic of the program, with the main flow of the program is obscured.

But not all programs are carefully written. Programmers get tired of all that checking, and tend to become lazy and to skip some tests that should really be done, leading to serious failures due to errors that a program could have avoided. In security-critical applications, such as network services, an uncaught error can result in a breach of security.

Some programming languages are equipped with a feature, called *exception handling*, that alleviates both the problem of a program becoming cluttered with rarely executed error-checking code and the problem of programmers failing to check for error conditions. With exception handling, code that is concerned with rarely occurring errors can be removed from the main part of a subprogram definition, making the logic of the program more apparent. Also, a program can be designed to recover from many kinds of errors, even errors that are made in untrustworthy and poorly written modules that contain no error checks in them.

Failure can also be exploited as part of a control mechanism. If you imagine trying to find your way through a maze, failure is hitting a dead end and recovery from failure is backing up and trying a different direction. We also explore this idea of *backtracking*, and compare it to exception handling.

Few languages support both exception handling and backtracking. Since Cinnameg does both, we use it for examples. Cinnameg's support for exception handling is similar to the same feature in several other languages.

16.2. Exception handling

Exception handling is provided by two related features. The first feature is the ability to cause the program to fail. When a program fails, it is said to cause, or to *raise* (or sometimes to *throw*), an exception. The second feature is the ability to *catch*, or to *handle*, an exception when one occurs. We begin by looking at how to catch an exception.

Catching exceptions

In Cinnameg, you can create a **Try**-statement that catches, or handles, exceptions generated within it. A Try-statement is similar to an If-statement and a Try*-expression is similar to an If*-expression. Expression

```
Try* A
  then B
  else C
%Try*
```

starts by evaluating A. If that evaluation is successful, then expression B is evaluated, and its value is the value of the Try*-expression. But if an exception is raised while evaluating A, then expression C is evaluated instead, and its value is the value of the Try*-expression. Expression C is the *exception handler*, which decides what to do to recover from an exception. In a Try-statement, either or both of phrases `then` B and `else` C can be omitted, both defaulting to "do nothing.")

Identifying exceptions

When status values are used instead of exceptions, there must be a separate test for each use of a function that returns a status value, and for each error that might occur. But with exception handling, an entire section of code, possibly involving several different events that might raise exceptions, can be covered by a single exception handler at the end of the section.

But in order to handle many possible errors, an exception handler usually needs to know what kind of error occurred. When a program fails, it raises some particular exception. For example, in Cinnameg, attempting to read an integer where there is no integer in the input raises exception *conversionX*. Division by 0 raises an exception of kind domainX, indicating that a value was passed to a function that the function does not allow, and having an associated string indicating the particulars. Specifically, dividing by 0 raises exception domainX("divide by zero"). An exception is a value that can be examined like other values. Figure 16.1 shows a program fragment that checks for conversionX or domainX.

Example

Imagine that you would like to compute and print the inner product of two vectors, where the inner product of $[a, b, c]$ and $[d, e, f]$ is $ad + be + cf$. You are only allowed to compute

```
Try
     section whose failure will be caught
  else
    Choose matching exception
      case conversionX => handle conversionX
      case domainX(s)  => handle domainX
      else             => handle other exceptions
    %Choose
%Try
```

Figure 16.1: **A Cinnameg fragment that catches an exception and tests the exception.** Reserved word **exception** stands for the exception that was raised when and if computation failed. Each case in the Choose-statement has a pattern that is matched against the exception.

the inner product of two vectors that have the same length; otherwise, the inner product is undefined. Figure 16.2 shows a function that prints the inner product of two vectors, or prints a message that the inner product is undefined. The exception handling code is at the end, and does not clutter the main logic.

16.2.1. Raising exceptions

In Cinnameg, you can raise exception e using expression `fail(`e`)`. At a failure, normal processing stops, and the system begins searching for an exception handler. Another way to cause failure using Cinnameg is to use an *assertion*. If E is a condition (an expression of type Boolean) and x is an exception, then `Ensure` E `else` x `%Ensure` does nothing if E is true, and raises exception x if E is false. For example,

```
Ensure lst =/= [] else emptyListX.
```

says only to continue this line of computation when lst is not empty. If you omit the **else** part, the exception is assumed to be standard exception testX. Assertions will come in useful in the next section, on backtracking.

16.2.2. When to use exception handling

Exception handling provides the possibility of an alternative control mechanism to the usual ones such as choices, loops and recursion, and it is possible to write a program that exploits exception handling as the normal mode of computing. For example, a program can read a list of numbers from a file until an attempt to read a number fails. You recover from the failure, recognizing it as the end of the file. That is not an error, of course, since you normally expect to see the end of the file eventually.

One school of thought is that this is an acceptable way to envision computation and to write programs. That is how Cinnameg handles multiple equations in a functional program. The compiler translates multiple equations into a program that tries the first equation inside a Try-expression. If that equation fails (for example, because its pattern match fails), the program catches that failure and tries the next equation, and so on. If

```
{case ip([],[])       = []
 case ip(x::xs, y::ys) = (x*y) :: ip(xs,ys)
 else ip(?)            = fail(sizeX)
}
{PrintInnerProduct(v1,v2). =
 Try
    {inner = ip(v1,v2)}
    Displayln "%(v1).%(v2) = %(inner)".
 else
    Displayln
      "Cannot compute the inner product of %(v1) and %(v2)".
 %Try
}
```

Figure 16.2: **Catching an exception in Cinnameg.** PrintInnerProduct prints the inner product of vectors v1 and v2, handling the case where they have different lengths by printing an error message. Notice that the error handling code is out of the way of the main logic. Function ip is written in an equational style, using recursion. There is no need for an initial check of the vector lengths. Instead, a length difference is detected deep inside the recursion of ip. Expression `fail sizeX` causes a failure with exception sizeX. Function $ converts a value to a string, for printing.

used carefully and clearly, that can lead to simple and elegant programs, and it is heavily used in logic programming.

On the other hand, overuse of exception handling, or use in awkward ways, can create programs that are truly difficult to understand. There is a second school of thought that exception handling should only be used to recover from situations that are not expected to happen during normal computation. That viewpoint is exemplified by Java, where you are discouraged from throwing exceptions unless something is really wrong. There are at least two reasons for that. First, the very idea of an exception is of something that is unusual, and that does not normally happen. Accordingly, you might argue that an exception should only occur under truly exceptional circumstances. Second, exception handling can be an unusually difficult control mechanism to understand. When an exception occurs, functions begin to be terminated abnormally, and control immediately transfers to a position that could be remote from the place where the exception was raised. Using exceptions as the usual control mechanism has the potential to yield programs that are, you might say, exceptionally difficult to understand.

Both schools of thought have their adherents. To help deal with the different approaches, Cinnameg has three kinds of exceptions. *Normal* exceptions are expected to occur during the normal course of program evaluation, and only those exceptions are caught when going through a list of cases. *Catchable* exceptions are only expected to occur in unusual circumstances. Finally *error* exceptions indicate mistakes in the program, and cause the program to stop.

```
{x = Backtrack* 1 else 2 %Backtrack*}
Ensure x > 1.
Displayln x.
```

Figure 16.3: **A program fragment that employs backtracking.** This fragment prints 2, since the choice of $x = 1$ cannot make it past the assertion that $x > 1$.

16.3. Backtracking

There is another approach to handling failure, called *backtracking*, that works differently from exception handling, and that is expected to be used as a control mechanism during normal computation rather than only for handling extraordinary circumstances. Cinnameg expression

`Backtrack*` A `else` B `%Backtrack*`

is evaluated by first computing expression A. If evaluation of A fails, then B is evaluated, and its value is used instead.

But suppose that evaluation of A succeeds. Then the value of the backtrack expression is *tentatively* taken to be A, and computation continues. But if the continuation of this computation, beyond the Backtrack-expression, fails, then it is as if A has failed, and the Backtrack-expression catches the failure and instead computes B as its value. For example, the program of Figure 16.3 prints 2. The first value $x = 1$ causes the Ensure statement to fail, raising exception testX. That causes computation to back up to the Backtrack*-expression and to try making $x = 2$ instead, a choice of that makes the assertion successful. Notice that the failure occurs after finishing the Backtrack*-expression, and yet is caught by that Backtrack*-expression.

Branching computations

Imagine that you are exploring a maze. When you enter the maze, you tie a string to a tree, and as you walk you unroll the string behind you. In your pocket you carry a supply of brightly colored pebbles. For simplicity, suppose the maze has no cycles, so you cannot walk forward and come back to where you were.

Suppose you come to a fork in the maze. Since you don't know which way to go, you choose a passage to explore, placing a pebble on the maze floor to show that you have tried that way. Eventually, you hit a dead end. When you do, you begin backing up, rolling up the string, until you return to the fork. Then you proceed down another passage, also marking it by placing a pebble on the floor.

Backtracking works similarly to exploring a maze. A Backtrack*-expression represents a fork in the maze. As the program runs, imagine it unrolling a string behind it, keeping track of where it has been. At a failure, the program backs up along the string, undoing its previous actions, until it gets to the fork. Then it tries the other branch. Figure 16.4 illustrate computation of the program in Figure 16.3 as a fork. A Backtrack*-expression is a branch point for the needle, or a fork in the maze. Figure 16.5 shows another example whose computation is diagrammed Figure 16.6.

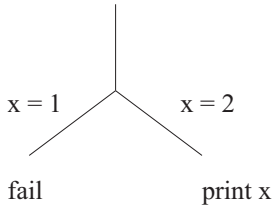


Figure 16.4: **A diagram showing computation of the program of Figure 16.3.** The fork is a branch point where the program can try either direction. Since the left hand branch leads to failure, the right-hand branch is tried instead.

```
{article = Backtrack "a" else "the" %Backtrack}
{noun = Backtrack "cat" else "mouse" %Backtrack}
Displayln "%(article) %(noun)".
Fail textX.
```

Figure 16.5: **A slightly more involved example of backtracking.** Figure 16.6 shows the branches that this computation takes. It prints four lines,

```
a cat
a mouse
the cat
the mouse
```

and then fails, backing up to look for prior branch points to explore.

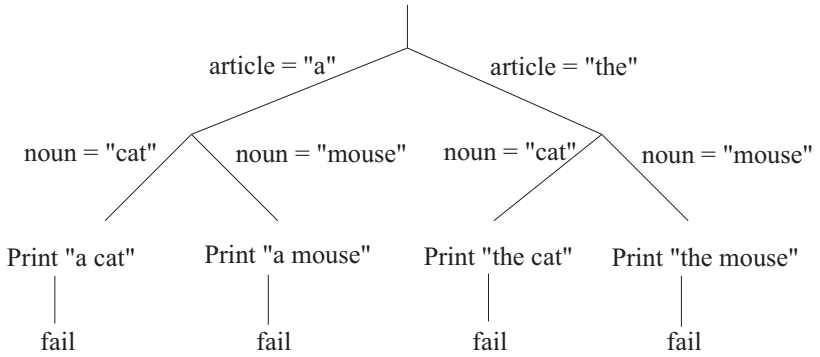


Figure 16.6: **Computation of the program in Figure 16.5.**

```
{farmer = 'f'; wolf = 'w'; goat = 'g'; cabbage = 'c'}
{somethingIsEaten(bank) =
  not(farmer `in` bank) and
  (goat `in` bank) and
  (cabbage `in` bank or wolf `in` bank)
}
{admissible (lft,rgt) =
  not(somethingIsEaten(lft)) and not(somethingIsEaten(rgt))
}
```

Figure 16.7: **Implementation of** *admissible***.** Expression admissible(s) yields true just when s is an admissible state for the farmer-wolf-goat-cabbage problem. The parameter of admissible is an ordered pair. Helper function somethingIsEaten checks one bank.

16.3.1. A backtracking example

Backtracking is useful for solving problems that involve searching or planning. An example is the problem of a farmer who owns a wolf, a goat and a cabbage. The farmer needs to get across a river in a boat, taking his possessions with him. The boat only has room for the farmer and one of his possessions. The animals will not run away, so it is acceptable to leave them alone. The difficulty is that, if the goat and the cabbage are left alone, the goat will eat the cabbage; and if the wolf and the goat are left alone, the wolf will eat the goat. How can the farmer cross the river without any of his possessions being eaten?

We will design a program that finds a solution based on a *generate-and-test* approach, where solutions are generated and explored by separate branches of the computation. Each branch checks itself to make sure that it is not exploring a disallowed solution (where, for example, some possession is being eaten). If a branch sees that something is wrong, it simply fails. That causes the program to backtrack and to try a different, possibly more promising, branch. Only those branches that are exploring acceptable solutions will make it to the end.

Cinnameg supports sets, which we will use for showing what or who is on each bank. Explicit sets are enclosed in {`:...:`}, such as {`:1,2,3:`}. Expression x `` `in` `` s is true if x is a member of set s;[1] $s +/x$ computes $s \cup \{x\}$ and $s -/x$ computes $s - \{x\}$,

States

A *state* tells who and what are on each bank of the river. For a state we will use a pair of sets (L,R), where set L tells what is on the left bank and set R tells what is on the right bank. For simplicity, we use characters 'f' for the farmer, 'w' for the wolf, 'g' for the goat and 'c' for the cabbage. The boat is always where the farmer is, so there is no need to show it. For example, ({'f','g'}, {'w','c'}) shows the farmer and goat on the left bank and the wolf and cabbage on the right bank. Say that a state is *admissible* if nothing is being eaten. Function admissible, shown in Figure 16.7, returns true on an admissible state, false on an inadmissible state.

[1] Binary operators that are words are surrounded by left-single quote marks in Cinnameg.

```
{next(lft,rgt) = st |
  open If farmer `in` lft then
    {possession = each lft} %% possession can be the farmer
    {st = (lft -/ farmer -/ possession, rgt +/ farmer +/ possession)}
  else
    {possession = each rgt}
    {st = (lft +/ farmer +/ possession, rgt -/ farmer -/ possession)}
  %If
}
```

Figure 16.8: **Implementation of function** *next***.** Expression next(s) returns a state that might follow state s in a solution of the farmer-wolf-goat-cabbage problem. It uses backtracking to produce a branch for each possible next state. Library function *each* takes a list as a parameter and backtracks over the members of that list, one branch returning each member. The farmer takes nothing when possession is selected to be the farmer. The word **open** allows the definition of st to be in scope outside the If-statement.

Finding the next state to try

There are up to four things that the farmer might conceivably do from a state; he might row across the river with any one of his three possessions, or he might row across alone. It is not our concern yet whether any of those actions leads to a solution, or even to an admissible state; that will be tested elsewhere. Right now, we only want to know what options the farmer has. Let next(s) produce, by backtracking, all possible states that might follow state s. (It produces only one of them in each branch, and can produce several branches.) For example, if state s is ({'f','w','g','c'},{ }), then computation of

```
{t = next(s)}
```

yields four branches

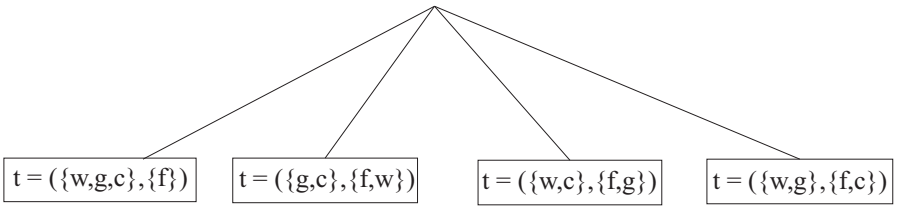


Figure 16.8 shows an implementation of the *next* function.

Building up a solution

A solution to the farmer-wolf-goat-cabbage problem can be described by a list of states, starting with the initial state (all on the left bank) and giving, in sequence, the states that occur as the farmer rows back and forth across the river. The list should end on a state where all are on the right bank. Getting from one state in the list to the next one should involve the farmer crossing the river once.

```
{addState(soln) = newState :: soln |
  {newState = next(head(soln))}
  Ensure admissible(newState).
  Ensure not (newState `in` soln).
}
```

Figure 16.9: **Implementation of function** *addState***.** The second Ensure line ensures that the new state is not one that has already been reached earlier. Operator 'in' is overloaded, and here tests whether a state is a member of a list rather than a member of a set.

It is convenient to build the solution up in reverse order, so that the more recently added states are near front of the list. Just read the lists backwards. At each point during the search, you have a partial solution that starts (at the end of the list) with the initial state ({'f','w','g','c'},{ }), and *might* be a good beginning to the full solution. The initial partial solution is list [({'f','w','g','c'}, { })], a list holding only the initial state. New states are added to the front of the list to explore slightly longer partial solutions.

To generate a solution, you start with an initial solution and extend it, ensuring after each extension that (1) the state just reached is admissible, and (2) the state just reached has not been reached before (to prevent looping). Function addState, shown in Figure 16.9, adds a state to a sequence of states, checking for consistency, and returning the slightly longer sequence of states. If the new state should not be tried, then addState fails.

Look at the structure of function addState. It is linear; there is no apparent looping or recursion. The body of addState just says to find the next state and move to it, as long as it is a good state. You are not concerned with the process of searching through all of the possible solutions, but only with describing what a correct solution looks like. Backtracking allows simple and elegant solutions to problems like this one. It allows you to linearize a computation that is, in reality, branching.

Finishing the solution

All that is needed to finish the farmer-wolf-goat-cabbage problem is a function that builds the entire solution. Function findsoln, shown in Figure 16.10, generates an extension of a given partial solution that reaches a given goal state, by adding as many states as necessary. In more detail, expression findsoln(partialPlan, goal) returns a list L such that head(L) = goal, partialPlan is a suffix of L, and L is a good plan in the sense that all of the states in L are admissible and all of the moves between consecutive states are allowed. (Function findsoln does not check whether partialPlan itself is good. It presumes that test was done earlier.) Now

```
Execute
  {everything = {:farmer, wolf, goat, cabbage:}}
  {soln = findsoln([(everything, {::})], ({::}, everything)).
  Displayln "Solution:  %(reverse soln)".
%Execute
```

finds a solution and writes it (in reverse order, since it was built up backwards.) An Execute block keeps running until all of its branches are done, so it will print all solutions.

```
{first
  case findsoln(goal::rest, goal) = goal::rest
  case findsoln(rest,goal)        = findsoln(addState(rest), goal)
}
```

Figure 16.10: **Implementation of function findsoln.** Expression findsoln(rest, goal) extends list rest by adding zero or more states to its beginning until the list begins with the goal state, and returns the final list, which is the plan. The first case applies when the list already begins with the goal. The second case uses addState to add one more state, ensuring that added states are admissible ones that are not already in the list.

16.3.2. Pattern matching in lists

One use of backtracking is in searching for patterns in lists (such as strings). Some languages, such as Icon, are based on pattern matching in strings. For example, suppose that you would like to check whether a string called *text* contains any of the substrings "the cat", "the mouse", "a cat" or "a mouse". Cinnameg pattern a ‘bor‘ b uses backtracking to try matching a, and if that fails, to match b. Using the string concatenation operator ++ in a pattern, the following will succeed if *text* contains any of the four listed strings.

```
{?  ++ ("the " ‘bor‘ "a ") ++ ("cat" ‘bor‘ "mouse") ++ ?  =~ text}
```

Matching pattern a ++ b against a target list L tries all possible ways of breaking up list L into two lists. For example, matching pattern a ++ b against list $[1, 2, 3]$ creates a computation with four branches. The first binds a to $[\,]$ and b to $[1, 2, 3]$, and the second binds a to $[1]$ and b to $[2, 3]$. All four possible values of a and b such that a ++ $b = [1, 2, 3]$ are tried.

The list membership testing function can be expressed as a pattern match. The idea is to try to do a match and to see whether it succeeds. We use Cinnameg expression ?# S #?, which yields true if statement S succeeds and false if it fails.

```
{member(m,x) =
  ?#
    {? ++ [m] ++ ? =~ x}
  #?
}
```

Suppose that, instead of testing whether m is a member of list x, you want to get a member of x, as function *each* does. Pattern matching can be used for that as well, yielding a definition of each whose structure is almost identitical to the pattern match in the membership testing function.

```
{each(x) = m |
  {? ++ [m] ++ ? =~ x}
}
```

The similarity exemplified here between (1) testing whether a given value has a given property and (2) finding something that has a given property (such as membership in a list) is exploited heavily in logic programming, the topic of Chapter 17.

16.3.3. Backtracking and substitution

Allowing an expression to produce more than one result, in different branches, comes at a cost in mathematical elegance. One of the rules of mathematics (and of pure functional programming) is that, if $A = B$, you should be able to replace A by B or B by A without changing what is computed. But suppose you write program

```
{x = Backtrack* 1 else 2 %Backtrack*}
{y = x + x}
```

That creates two branches, one where $y = 2$, the other where $y = 4$. But, replacing x with the expression that computes it yields

```
{y = Backtrack* 1 else 2 %Backtrack* +
     Backtrack* 1 else 2 %Backtrack*}
```

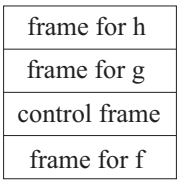
which produces four branches, with values $y = 2$, $y = 3$, $y = 3$ (again) and $y = 4$. Logic programming allows a language implementation to perform automatic backtracking but restores the correctness of some elementary forms of mathematical reasoning.

16.4. Mechanics of handling failure

16.4.1. Implementation of exception handling

When a subprogram is called, a frame for it is pushed onto the run-time stack. Similarly, when a try expression is entered, a *control frame* is pushed onto the run-time stack, containing information about where to resume when an exception is raised.

When the program raises an exception, an *exception manager* runs. It searching down the run-time stack for the topmost control frame, discarding any subprogram frames that it encounters (causing those subprograms to stop immediately). When a control frame is encountered, it is removed, and computation resumes with the exception handler that is indicated by the control frame. For example, suppose that function f contains a Try*-expression. Inside the try, f calls g, which in turn calls h. The run-time stack is as follows.



Now suppose h raises an exception. The result is that the frames for h and g are removed, as is the control frame, and f resumes at the exception handler for the try expression, whose location is remembered in the control frame.

A program can enter a try while it is inside another try. In that case, the run-time stack contains more than one control frame. When the program raises an exception, the topmost control frame, belonging to the most recently entered try, is the one that is found. Try*-expressions are nested *dynamically*, not statically.

Of course, computation normally does not fail, and all goes well. When the body of a Try-statement succeeds, the control frame is removed without invoking the exception handler.

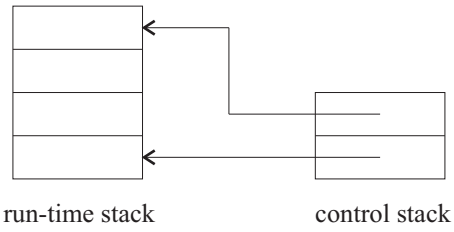


Figure 16.11: **Placing control frames in a separate control stack.** Notice that each frame in the control stack indicates where it belongs in the run-time stack.

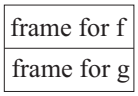
16.4.2. Implementation of backtracking

Exception handling can be implemented by putting control frames in the run-time stack. But to deal with backtracking, we will need to make a modification to how control frames are handled. Instead of putting them in the run-time stack, we put them in a separate *control stack*. Each control frame points to the frame in the run-time stack that created it. The two stacks (the control stack and the run-time stack) are pictured in Figure 16.11. When a failure occurs, the top frame on the control stack tells where to continue execution.

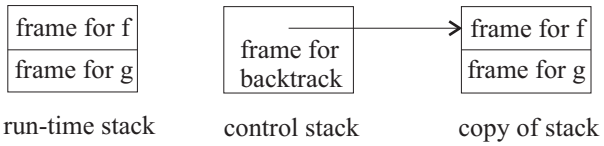
A Backtrack expression differs from a try in the following important way. When expression A succeeds in `Backtrack*` A `else` B `%Backtrack*`, the control frame is not removed from the control stack. Instead, it is kept there, ready to catch an exception when one is raised, and to resume by computing B. That leads to a serious implementation problem. The control frame points to a frame in the run-time stack. Since the control frame is not removed when the Backtrack*-expression is exited, it can continue pointing to a frame even after the subprogram that owned that frame has returned. Because of that, it appears not to be possible to remove frames from the run-time stack. That can be avoided by making a control frame point to a copy of the run-time stack. For example, consider the following.

```
{f() = Backtrack 1 else 2 %Backtrack}
{g() =
  {x = f()}
  Displayln(x).
  Fail testX.
}
```

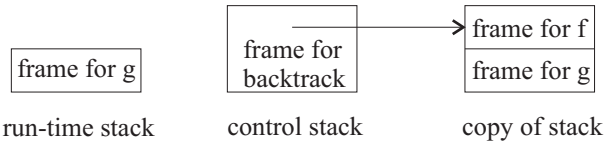
Imagine computing $g(\,)$. The first thing g does is evaluate $f(\,)$. When $f(\,)$ begins to run, the run-time stack looks as follows, with the frame for f on top of the frame for g.



When $f(\,)$ enters the backtrack expression, a control frame is pushed onto the control stack, and the situation changes to the following.



Notice that the entire run-time stack has been copied, and the control frame points to the copy. Now the backtrack construct finishes, and $f(\,)$ returns with value 1. The situation looks like this.



Now $g(\,)$ prints the value 1, and then fails. When a failure occurs, the top control frame on the control stack tells where to resume computation. In the example, the control frame indicates that computation should resume in the Backtrack*-expression, at the point where it produces the alternative value 2. The entire run-time stack is thrown out and replaced by the copy that is remembered with the control frame. That restores the run-time stack to the point where it was when the Backtrack*-expression was started, and $f(\,)$ can resume computing. This time, $f(\,)$ returns 2.

Copying the entire run-time stack is expensive, and it can be avoided if some care is taken. All that is really needed is an implementation that works *as if* it copies the stack. The idea is to copy parts of the stack only when they genuinely need to be copied. One way to do that is to store the run-time stack as a linked list of frames, letting the dynamic link (Section 8.2.2) be the list link. Using reference counts (Section 8.1.4), you can ensure that a frame is not destroyed or modified prematurely. If you try to modify the information in a frame that has more than one reference to it, you need to copy just that frame first.

16.4.3. Pruning the search using commits

Introduction of a backtrack control frame leads to copying of at least part of the run-time stack. The program needs to remember every place that it has performed a branch, just in case there is a need later to back up. That can result in a vast amount of memory being used, and can be a serious problem for programs that employ backtracking.

Sometimes, a program gets to a point where it knows it will not backtrack, or it knows that any backtracking that it would do would not lead to any useful or desired computations. In that case, the program should cause the backtrack control frames to be removed, and the memory that they are using to be recovered. The operation that does that is called a *cut* or a *commit*. We will use the term *commit*, and describe how commits are done in Cinnameg.

There are two parts to the implementation of a commit. The first part concerns management of *marker frames*. Sometime before the commit is done, a marker frame is pushed onto the run-time stack, holding a pointer to the current position of the top of the control stack. In Cinnameg, CommitBarrier constructs control pushing and popping of marker frames. On entry to `CommitBarrier ... %CommitBarrier`, a marker frame is pushed, and on exit from that construct, the marker frame is removed. But keep in mind that creating a branch at a backtrack construct copies the run-time stack. Exiting `CommitBarrier ...`

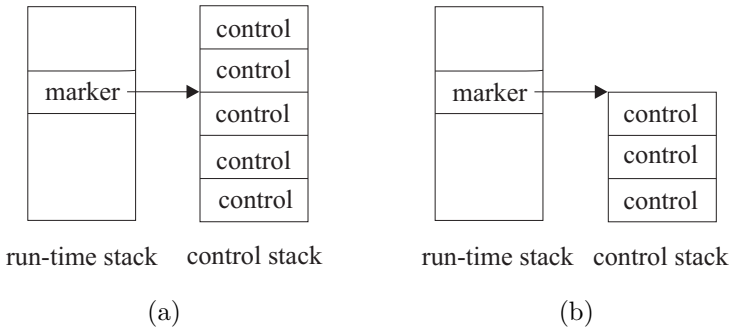


Figure 16.12: **The action of a commit.** The run-time stack before a commit (a) holds a marker frame pointing to a position in the control stack. The commit removes all backtrack frames above the position indicated by the topmost marker frame (b).

```
Execute
  CommitBarrier
    {everything = {:farmer,wolf,goat,cabbage:}}
    {soln = findsoln([(everything, {::})],
                     ({::}, everything))}
    Commit.
    Displayln "Solution:  %(reverse soln)".
  %CommitBarrier
%Execute
```

Figure 16.13: **A program that prints only the first solution that is found to the farmer-wolf-goat-cabbage problem.** It uses function findsoln from Figure 16.10. Once a solution is found, other solutions are ignored by removing the backtrack control frames that have been pushed onto the control stack while running findsoln.

`%CommitBarrier` only removes the marker frame from the currently running copy of the run-time stack.

The next part is the commit itself. Statement `Commit.` tells the interpreter to find the top marker frame in the current run-time stack and to remove all backtrack control frames that are above the indicated position in the control stack. Figure 16.12 shows the action of a commit.

The farmer-wolf-goat-cabbage problem has two solutions. Having found the first one, suppose that you want to ignore the other one. Figure 16.13 shows a way of doing that by removing the backtrack control frames that would lead to finding the second solution.

16.5. Variables and branching computations

Suppose a computation forks into two branches. Since backtracking involves making a copy of the run-time stack, anything that one branch of the computation does to the run-time stack, such as changing the value of a local variable, does not affect any other branch.

```
{/:nonshared:/ @z = "ben"}
Backtrack
  {@z =! @z ++ "d"}
else
  {@z =! @z ++ "t"}
%Backtrack
Displayl @z.
(fail testX)
```

Figure 16.14: **An Cinnameg program using a nonshared box.** This program prints bend and bent on separate lines and then fails. The two branches do not interfere with one another.

But there can be other variables, called boxes in Cinnameg, that live longer than a single subprogram execution. A box is an object that can hold one value. Cinnameg lets you create, initialize and name a box b using statement {`@`b `=` i}; and to replace the value in box b by v using statement {`@`b `=!` v}. Expression @b stands for the value currently stored in box b. For backtracking, boxes present a problem: if one branch changes the content of a box, then it might interfere with what another branch is doing. For example, program fragment

```
Try
  {@b = 10}
  {x = Backtrack 1 else 2 %Backtrack}
  {@b =! @b + 1}
  Displayln (x, @b).
  Ensure false.
%Try
```

prints (1,11) and (2,12) since the first branch changes what is in box b. The second branch's failure is caught by the Try.

There are times when sharing boxes is desirable, as when you use a counter to count how many different branches the program explores. But most of the time, you would prefer for the second branch to behave exactly as it would have behaved if the first branch had not been run at all. We really need two different kinds of boxes: *shared* boxes, which are shared by different branches, and allow one branch to affect what another does; and *nonshared* boxes, which behave as if the other branches did not run at all. Cinnameg provides both kinds of box. By default, a box created by {`@b`} is shared. But if you create a box n via {`/:nonshared:/ @n`}, then you get a box that is private to each branch. Any change made to the content of this box in one branch is not seen by parallel branches. Figure 16.14 shows a program that uses nonshared boxes.

16.5.1. Implementation of nonshared boxes

A typical way to implement nonshared boxes is to store *trail* frames in the control stack. Each trail frame holds a box and its content, and any time a nonshared box has its content changed, that box and its old content are pushed on the control stack. When the program

backtracks, it restores the old contents of the boxes found in the trail frames that are above the topmost control frame, making it appear that the box was not changed.

16.6. Exercises

16.1. Explain how exception handling can make it possible to run an unreliable function (that might cause errors) without the worry that those errors will cause the entire program to stop running. Assume that the function does not run forever.

16.2. Java allows a function to produce a special **null** value, indicating nothing, or no result. Instead of adding exception handling, why doesn't Java just let you use a null result to indicate an error? Does exception handling have any advantages over that?

16.3. Write a Cinnameg function that tries to read an integer from the standard input and to return that integer, but that produces answer -1 if no integer could be found. Library function readInteger( ) returns an Integer. It will read an integer from the standard input and return it, but will fail with exception conversionX if there is no integer to read.

16.4. Write a Cinnameg function that reads several integers from the standard input. It should continue reading until there are no more integers, and return a list of all of the integers that were read.

16.5. Write a Cinnameg statement that binds x to $f(y)$ provided $f(y)$ succeeds, and binds x to 0 otherwise. Be careful about scoping. The binding of x should be visible after your statement.

16.6. The map function, described in Chapter 10, is defined so that map f [a, b, c] = $[f(a), f(b), f(c)]$. Write a version of map called mapIgnoreFail that can handle cases where evaluation of f fails, putting only results that succeed in the result list. For example, if $f(a)$ and $f(c)$ succeed but $f(b)$ fails, then mapIgnoreFail f [a, b, c] = $[f(a), f(c)]$.

16.7. Common Lisp has an exception handling mechanism that is roughly as follows. Expression (catch X E) sets up an exception handler that can handle exception X, and evaluates E with this exception handler in place. Expression (throw X V) is evaluated by searching for an active catch-expression that will catch exception X, and causing the catch-expression to return V as its value. Ignoring syntactic details, how does this differ from the approach of using a Try*-expression. Who decides what happens when the exeption occurs?

16.8. Backtracking can be simulated using recursion. Modify the solution to the Farmer-Wolf-Goat-Cabbage problem so that findsoln yields a list of all solutions, and does not employ direct backtracking.

16.9. If a language does not support nonshared boxes, can they be simulated? If so, how?

16.10. The knight's tour problem asks how a knight placed in one square of an $n \times n$ chessboard can move around the board, hitting each square exactly once and returning to the start square. Write a solution to the knight's tour problem in Cinnameg using backtracking.

16.11. The knight's tour problem described in the preceding exercise can be solved efficiently without backtracking. A square is available if the knight has not visited it yet. At each step, the knight moves to an available square that has the most moves to available squares after that. Implement this algorithm, and compare it to the backtracking algorithm for speed.

16.12. Three missionaries and three cannibals are on one bank of a river. They want to get across to the other bank. They have a boat that can hold up to two people at a time. To go across the river, the boat must have at least one person in it to paddle. The paddler can be a missionary or a cannibal. The difficulty is that, if at any time the cannibals outnumber the missionaries on either bank, they will eat the missionaries. (A missionary cannot hide in the boat. You are on the bank even if in the boat on that bank.) Write a program that finds all ways that all six people can cross the river without anyone being eaten, and that do not go through any state more than once.

16.13. What is the key difference between the implementations of try control frames and backtrack control frames? Ignore where they are placed. Discuss the timing of adding and removing them.

16.14. In logic programming, variables are like nonshared boxes, but a variable can have no value (or be an empty box). Only variables without a current value can be changed; once a variable has a value in a given branch of the computation, the value is permanent for that branch. How does that simplify trail frames?

16.15. Imagine a language, such as Cinnameg, that allows both exception handling and backtracking. Derive a way to manage the two together. Explain how you intend to keep track of control information, how you find where to pick up after a failure, and how control frames are added and removed both for exception handling and for backtracking. Control frames can be tagged as being either for backtracking or for exception handling, and they can contain extra information.

16.16. When are commits most important in backtracking computations? If you view a branching computation as a tree, are commits more important for computations that are wide but shallow trees, or for computations that are narrow but deep trees?

16.17. Commits can save both memory and time.

(a) Explain how commits save memory.

(b) Explain how commits save time.

16.18. Can you see any danger in using commits? What surprising problems might emerge with their use?

16.7. Bibliographic notes

Goodenough [40] introduces some ideas on exception handling.

SNOBOL [45] is an early language that employed backtracking. Icon [46, 47] is a more recent language in the SNOBOL family. Logic programming languages, such as Prolog [28, 29], also employ backtracking to control computation.

Chapter 17

Logic Programming

17.1. What is logic programming?

Although backtracking (Chapter 16) is a powerful control mechanism, it can be somewhat difficult to understand, and convincing yourself that a program that employs backtracking is correct can be difficult. Logic programming is a form of declarative programming that is partly a way of putting reins on backtracking, making it more manageable. Where, in a functional program, facts are expressed as equations, a logic program might express "facts" such as "All apples are good to eat" or "If I can see Orion, it must be Winter." To convince yourself that a logic program can only produce correct answers, it suffices to make sure that your stated facts are, indeed, facts.

A logic programming implementation consists of an *inference engine*, typically an interpreter, that is used to perform logical inferences of new facts from old. It converts your facts into an algorithm that finds proofs, based on the given facts, and it is the action of this engine that is responsible for the computation of a logic programming system.

There are several forms of logic programming, some with relatively simple engines and some with very sophisticated engines. We will focus on just one form of logic programming, exemplified by the language Prolog. It is the first form of logic programming that was implemented, and is still a popular one today.

17.2. Clauses and formulas

A *logical formula*, or simply a *formula*, is an expression that is either true or false. This section explores the structure of formulas.

Predicates and atoms

A predicate is a function that returns a boolean result. It captures properties of things or relationships among things. When you say that an apple is green, you capture the notion of "greenness" in a predicate, saying that green(apple) is true. The relationship "A is the mother of B" can be expressed as mother(A, B), where predicate mother is true of an ordered pair (A, B) just when A is the mother of B. A predicate can have any number of parameters, so that it can express complex relationships among several things.

An *atom* is an application of a predicate to an argument or arguments. For example, mother(X,Y) is an atom. The arguments can be constants or variables, or possibly more complex expressions that are discussed later. For example, if *greater* is a predicate, then greater(5, 2) is an atom. In some cases, a predicate has no arguments, and the predicate by itself is an atom.

Boolean operations and literals

Atoms can be combined using boolean operations. The usual boolean operations are as follows.

1. **Conjunction.** $A \wedge B$ is read "A and B", and is true just when both A and B are true.
2. **Disjunction.** $A \vee B$ is read "A or B", and is true when either A or B or both are true.
3. **Negation.** $\neg A$ read "not A", and is true just when A is false.
4. **Implication.** $A \Rightarrow B$ is equivalent to $(\neg A) \vee B$. Notice that, if A is true, then $A \Rightarrow B$ can only be true if B is also true. $A \Rightarrow B$ is read "A implies B",

The usual rules of precedence give $\neg$ the highest precedence, followed by $\wedge$, then $\vee$, with $\Rightarrow$ having the lowest precedence.

A *literal* is either an atom or the negation of an atom. For example green(apple) and ¬green(apple) are literals. A literal such as green(apple) that is not negated is called a *positive literal*, and a literal such as ¬green(apple) is called a *negative literal*.

Formulas and Horn clauses

Logical *formulas* are built by using boolean operations on atoms and other formulas in arbitrary ways. For example, $p(X, Y) \Rightarrow r(X) \vee q(Y)$ is a formula. But logic programming does not work with arbitrary formulas. Instead, it uses restricted kinds of formulas called Horn clauses that make the inference engine easier to define and easier to understand.

A *clause* is a disjunction of literals. That is, it has the form $L_1 \vee L_2 \vee \ldots \vee L_n$, for some $n \geq 1$, where each of $L_1, \ldots, L_n$ is a literal. (A literal by itself is considered to be a clause of one literal.) A *Horn clause* is a clause that contains exactly one positive literal. For example, suppose that mother and parent are predicates. Then ¬mother(X,Y) $\vee$ parent(X,Y) is a Horn clause, since it has exactly one positive literal. The positive literal is called the *head* of the Horn clause.

A Horn clause is not required to have any negative literals. A clause consisting of nothing but one positive literal states a basic fact. For example, mother(betty,bob) states the fact that betty is the mother of bob. If a Horn clause has at least one negative literal, then it can be expressed as an implication with just its head on the right-hand side. For example,

1. $\neg A \vee B$ is equivalent to $A \Rightarrow B$, from the definition of implication.
2. $\neg A \vee \neg B \vee C$ is equivalent to $(A \wedge B) \Rightarrow C$. That is, C is true if both A and B are true. You can understand why by imagining the case where both A and B are true, and asking how $\neg A \vee \neg B \vee C$ might also be true. Since neither $\neg A$ nor $\neg B$ is true,

the only way to make $\neg A \vee \neg B \vee C$ true is to make C true. That is, whenever A and B are both true, C must be true.

3. $\neg A \vee \neg B \vee \neg C \vee D$ is equivalent to $(A \wedge B \wedge C) \Rightarrow D$.

When a Horn clause is expressed as an implication, all of the literals become positive.

Horn clause $(A \wedge B) \Rightarrow C$ can be thought of as saying "if you want to show that C is true, try proving that both A and B are true." Logic programming is based on proofs; proving something and computing it are the same thing. In programming languages, it is common to write what you want to compute on the left hand side of a fact and to write information on how to compute it on the right hand side. For example, equation $f(x) = x*x$ tells you to compute $f(x)$ by computing expression $x * x$. To keep this familiar order, it is common practice to write implications using the right-to-left implication symbol $\Leftarrow$, where $B \Leftarrow A$ is the same as $A \Rightarrow B$.

Also, in a Horn clause, the $\wedge$ operator is usually replaced by a comma. So $C \Leftarrow A, B$ says, to prove that C is true, try proving that both A and B are true. The remainder of this section uses the modified notation for Horn clauses.

Programs

A logic program is a collection of Horn clauses, called the *axioms* of the program, stating facts that the inference engine can use. A fact that contains variables is understood to be true for all values of the variables. Computation of a logic program is the process of the inference engine deriving new facts from the axioms stated in the program.

17.3. Logic programs and computation

This section begins to look at how the inference engine works.

Goals

A *goal* is a question that you ask the inference engine to answer. The simplest kind of question is a yes/no question; you ask the engine whether it can prove a particular atom. In general, a goal is a list of atoms, and it is the inference engine's responsiblity to prove *all* of the atoms in the list. Each atom in a goal list is called a *subgoal*.

For example, if the goal list holds (parent(betty, bob), mother(betty, ron)), then the engine needs to find a way to prove both that parent(betty,bob) and mother(betty, ron) are true. Think of the goal list as a to-do list of things that need to be proved. The first thing that the inference engine does is select one of the atoms to work on. The simplest rule is to select the first subgoal in the list, and that is the rule that we will use.

Having selected a subgoal to work on, such as parent(betty, bob), the engine looks for a way to prove that subgoal. It examines the axioms, selecting all axioms that *might* work. An axiom is interpreted as follows for the purposes of proving an atom.

1. A clause that contains no negative literals states a fundamental fact. For example, if the engine is trying to prove that mother(betty, ron) is true, and mother(betty, ron) is stated as an axiom in the program, then the job is done. In this case, the subgoal is erased from the list of subgoals and the engine selects another subgoal to work on.

2. A Horn clause of the form $B \Leftarrow A$ says that, if you want to prove B, try proving A. That is, it suggests replacing subgoal B by subgoal A. For example, if the engine is currently trying to prove parent(betty, bob), and one of the axioms is parent(betty, bob) $\Leftarrow$ mother(betty, bob), then the engine can choose to replace subgoal parent(betty, bob) by subgoal mother(betty, bob). The engine modifies its list of subgoals, and again selects a subgoal to work on.

   When the engine removes a subgoal from its list and replaces that subgoal by another one, it needs to make a choice about where to put the newly inserted subgoal — should it go at the front of the list or at the back? The usual rule it to put the new subgoal at the front of the list. That way, after replacing parent(betty, bob) by mother(betty, bob), the engine will select mother(betty, bob) next to work on, continuing on this same line of reasoning.

3. A Horn clause of the form $(C \Leftarrow A, B)$ says that, if you want to prove C, try proving both A and B. So you remove C from the list of subgoals and insert A and B. In general, if there are n literals on the right-hand side of $\Leftarrow$, then you add all n of those literals to the subgoal list. The usual rule is to insert them so that they end up in the same order as they are in the axiom. When axiom $(C \Leftarrow A, B)$ is used to prove C, subgoal A will end up at the front of the new subgoal list, and it will be the next subgoal worked on.

Variables and pattern matching

When evaluating an expression in an equational program, pattern matching has to be used when performing substitutions. The reason is that the equations that a program states are more general than you need at a particular step in the computation. For example, if an equation states that $f(x) = x * x$, and you want to compute $f(3)$, then you need to find a special form, $f(3) = 3 * 3$, of the more general equation, and pattern matching does that.

Pattern matching is also used when evaluating a subgoal in a logic program, for the same reason. It needs to find a specialized form of an axiom that is suitable for a particular step in reasoning. For example, suppose that the program states, as an axiom,

$$\text{parent}(X, Y) \Leftarrow \text{mother}(X, Y)$$

When an axiom contains variables, the programmer is asserting that the axiom is true for *all* values of those variables. So, for any X and Y, if X is Y's mother, then X is also Y's parent. To prove goal parent(betty, bob), a pattern match is done, matching parent(X,Y) against parent(betty,bob). Clearly, X must be betty and Y must be bob. The specialized axiom is

$$\text{parent}(\text{betty}, \text{bob}) \Leftarrow \text{mother}(\text{betty}, \text{bob})$$

and subgoal parent(betty, bob) is therefore replaced by new subgoal mother(betty, bob). More details of pattern matching are discussed below.

The role of backtracking

Suppose that a logic program contains two axioms,

$$\begin{aligned} \text{parent}(X, Y) &\Leftarrow \text{mother}(X, Y) \\ \text{parent}(X, Y) &\Leftarrow \text{father}(X, Y) \end{aligned}$$

1. mother(marcy, cathy).
2. mother(sally, bertrand).
3. father(bertrand, marcy).
4. father(william, cathy).
5. father(jordan, william).
6. parent(X,Y) $\Leftarrow$ mother(X,Y).
7. parent(X,Y) $\Leftarrow$ father(X,Y).

Figure 17.1: **A sample logic program.** Each line is a Horn clause. Predicate mother(X,Y) means "X is the mother of Y," father(X, Y) means "X is the father of Y" and parent(X, Y) means "X is a parent of Y."

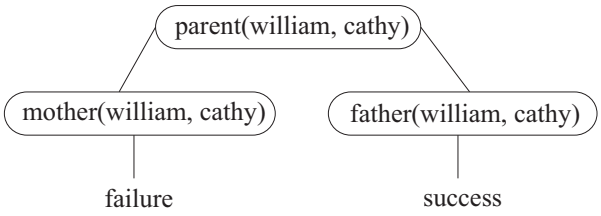


Figure 17.2: **Computation of goal parent(william, cathy), using the axioms from Figure 17.1.** The failure of mother(william, cathy) is because there is no Horn clause whose head unifies with mother(william, cathy). Goal father(william, cathy) is replaced by no goals, so the computation terminates successfully.

where the first says that X is a parent of Y if X is the mother of Y, and the second says that X is a parent of Y if X is the father of Y. A goal is to prove parent(a,b) for two particular individuals a and b. There are two ways to do the proof; you can either replace goal parent(a,b) by new goal mother(a,b), or you can replace it by new goal father(a,b). It is not clear which axiom to use. Certainly, you do *not* want to replace it by both new subgoals, putting both into the list of subgoals to be proved. That would say that, to prove that a is a parent of b, you must prove *both* that a is b's mother and that a is b's father. But what you want to do is to prove *either* that a is b's mother or that a is b's father.

A solution is to use backtracking. Each axiom is tried, but each is tried in a separate branch of the computation. If the first branch succeeds (so proving mother(a,b) succeeds), then parent(a,b) is proved. But if the first branch fails, then the second branch, which tries to demonstrate father(a,b), is tried instead.

Example

Figure 17.2 shows computation of goal parent(william, cathy), using the axioms shown in Figure 17.1. Each node of the tree contains a "to do" list of subgoals to be proved. The computation is shown as a tree, since it uses backtracking; each branch of the tree represents one way in which the goals might be proved. Notice that subgoal parent(william, cathy) is replaced by mother(william, cathy) in one branch and by father(william, cathy) in the other branch. *Failure* indicates that none of the axioms can be used to prove the first subgoal.

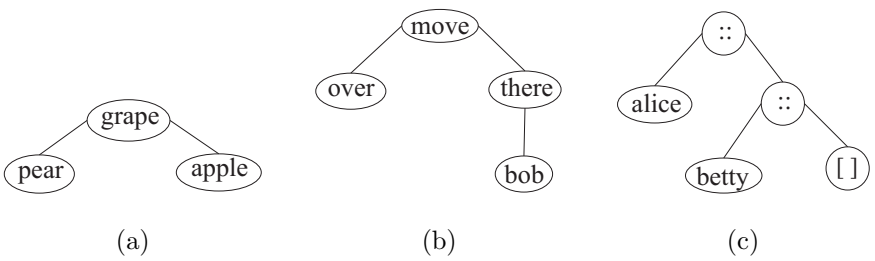


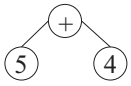
Figure 17.3: **Three trees that can be values in a logic program.** We allow :: as a symbol here. Tree (a) is written grape(pear,apple) and tree (b) is written move(over,there(bob)). Binary operator syntax, if supported by a language, allows you to write x::y instead of ::(x,y). So tree (c) is written alice::betty::[ ]. Syntax [alice, betty] is an abbreviation for alice::betty::[ ].

Success indicates that the list of subgoals is empty; all have been proved.

17.4. Trees and terms

Data values are trees. where the nonleaf nodes are labeled by symbols and the leaf nodes can be labeled by any kind of value that the language supports (a symbol, a number, a string, etc.). Figure 17.3 shows examples of trees. Lists are considered to be special cases of trees whose internal nodes are labeled by a list construction symbol (which we will call :: here), as is shown in Figure 17.3(b). The end of a list is marked by another symbol, often simply called [ ], since it is really the empty list. (So the sequence of two characters "[]" is another special name for a symbol.)

It is awkward to draw trees in programs, so you write them using an expression notation, and an expression that describes a tree is called a *term*. Term $r(a, b, c)$ stands for a tree whose root is labeled r, having three subtrees a, b, and c. The order of the subtrees is important; they are given from left to right. A leaf having label q is just written q, without showing any subtrees. When a node label is a binary operator, it is more convenient to use binary operator syntax. Expression $5 + 4$ stands for the tree



as if it were written $+(5, 4)$.

17.5. Unknowns

A logic program works by finding values for unknown quantities. For example, to find the factorial 6!, you state that the unknown quantity X is equal to 6! and let the program work from the facts that you have provided to determine the value of X. One of the basic concepts of logic programming is, therefore, a variable, or an *unknown*, representing some value that has not yet been determined.

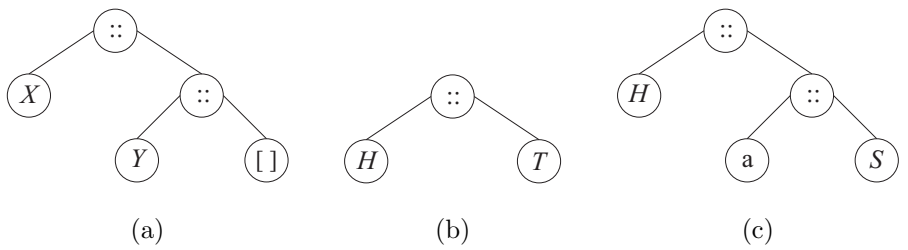


Figure 17.4: **Three trees containing unknowns.** Tree (a) is written $[X, Y]$, a list of two unknown things. Tree (b) is $H :: T$, a list with an unknown head and an unknown tail. If, later, unknown T becomes known to be $a :: S$, then tree (b) will be changed into tree (c), representing the extra knowledge about T.

An unknown starts life having no known value; it is truly unknown. At some point during computation information might be obtained that gives a value to the unknown. (So it becomes known, or *bound*.) An unknown differs from a variable in an imperative program in two key respects.

1. Once a value is given to the unknown in a given branch of a computation, that value cannot be changed in that same branch. The unknown is permanently bound. That makes logical sense; once you know the value, you need to stick with what you say, and cannot change your mind later (in the same branch).

2. An unknown can be bound to another unknown. If X is bound to Y, it means that binding either of X or Y also binds the other to the same value. They have become the same unknown.

We will adopt the convention of using a name that starts with an upper case letter for an unknown. For example, X and Who are unknowns. But be careful. An unknown can become bound shortly after it is created. It is still referred to by its name, such as X, but after becoming bound it is no longer unknown at all.

Unknowns in trees

A leaf of a tree is allowed to be an unknown, indicating an unknown subtree. Figure 17.4(a) shows examples. As unknowns become bound, more of the tree becomes known.

17.6. Unification

Since pattern matching is a key step in making use of an axiom, we need a pattern matching tool that can be used when the data values themselves contain unknowns that might be bound by the pattern match. That tool is *unification*.

Section 13.4.3 discusses unification and uses it to solve systems of equations involving types. Unification is able to solve equations over any *free algebra*, which is a domain where two terms can only be equal if they are identical terms. In a free algebra, equation $f(x) = f(y)$ can only be true if $x = y$, and equation $f(x) = g(y)$ can never be equal. The ordinary

algebra of numbers that you are familiar with is not free. For example, in the nonfree algebra of numbers, terms 1+2 and 2+1 are equal. In a free algebra, they must be different.

But two trees are only equal when they have exactly the same structure. When you think of terms as describing trees, then two terms are equal just when they are the same term. So, as trees, 1+2 (a tree whose left subtree is 1) is not equal to 2+1 (a tree whose left subtree is 2). So unification should work on trees.

Unification tries to find a way to bind unknowns so that two given terms are equal. The idea of unification is fairly simple. To unify terms $f(a, b)$ and $f(c, d)$, you cleary unify a with c and unify b with d. Unification of $f(a)$ with $g(b)$ fails — there is no solution. To unify an unknown X with a term t, bind X to t. (If T is another unknown, you end up binding one unknown to another without choosing a value for either one. The special case of unifying X with itself does nothing.)

A complicating factor is the *occur check*. Suppose that you want to unify X with $f(X)$. Then you bind unknown X to $f(X)$. But what term has X become? If X is bound, then every occurrence of it is bound, so $f(X)$ must become $f(f(X))$, which in turn becomes $f(f(f(X)))$. You never get rid of X, and the tree that you have derived is infinite, $f(f(f(f(\ldots))))$. You can avoid creating infinite trees by refusing to bind unknown X to any term that contains X (with the obvious special case that $X = X$ is true). So, by the occur check, equation $X = f(X)$ has no solution.

17.6.1. A careful treatment of unification

A *substitution* is a collection of variable bindings. We will write substitutions in braces. For example, $\{X = \text{joe}, Y = \text{table(chair)}\}$ is a substitution that binds variable X to joe and Y to the term table(chair). We will use Greek letters such as θ and α to name substitutions. If θ is a substitution and A is a term, write $\theta(A)$ for the term that results by performing substitution θ on term A, replacing each variable that is bound in θ by the term to which it is bound. For example, if A is term flower(Y, rose) and $\theta = \{Y = \text{four}\}$, then $\theta(A)$ is term flower(four, rose).

Suppose that $\phi = \{X = Y, Y = \text{eighty}\}$ is a substitution. Then $\phi(\text{five}(X, Y))$ yields $\text{five}(Y, \text{eighty})$. But that still contains variable Y, which ϕ says should be replaced by eighty. You could use ϕ again, but it is better to use a substitution that only needs to be performed once. Say that a substitution θ is *proper* if no variable that occurs on the left-hand side of an equation in θ also occurs on the right-hand side of an equation. Substitution ϕ is not proper, since Y occurs on both the left-hand side and the right-hand side of an equation. But substitution $\{X = \text{eighty}, Y = \text{eighty}\}$ is proper. If θ is a proper substitution then $\theta(\theta(A)) = \theta(A)$. The second application of θ cannot have any effect.

A proper substitution θ is a *unifier* for terms A and B if $\theta(A) = \theta(B)$. The goal of unification is to find a unifier for two terms. For example, $\theta = \{X = \text{eighty}, Y = \text{eighty}\}$ is a unifier for terms five(X, eighty) and five(eighty, Y). Say that terms A and B are *unifiable* if a unifier for them exists.

Principal unification

An important feature of unification concerns how variables are unified with one another. Suppose that you want to unify terms bird(X, Y) and bird(W, Z). You must ensure that X and W are the same, and that Y and Z are the same. There is a myriad of ways to do that. For example, to ensure that X and W are the same, you could bind each of them to symbol

$$
\begin{array}{rcl}
\text{unify}(v, v) & = & [\,] \\
\text{unify}(v, A) & = & [(v, A)] \text{ provided } v \text{ does not occur in } A \\
\text{unify}(A, v) & = & \text{unify}(v, A) \\
\text{unify}(f(), f()) & = & [\,] \\
\text{unify}(\ f(A_1, \ldots, A_n), & = & \theta_1 +\!+ \theta_2 \text{ where} \\
\quad f(B_1, \ldots, B_n)) & & \quad \theta_1 = \text{unify}(A_1, B_1) \\
& & \quad \theta_2 = \text{unify}(\ f(\theta_1(A_2), \ldots, \theta_1(A_n)), \\
& & \qquad\qquad f(\theta_1(B_2), \ldots, \theta_1(B_n)))
\end{array}
$$

Figure 17.5: **An equational definition of principal unification.** Given two terms S and T, unify(S, T) produces a principal unifier of S and T as a list of pairs $[(v_1, t_1), \ldots, (v_m, t_m)]$, indicating substitution $\{v_1 = t_1, \ldots, v_m = t_m\}$, where each v_i is a variable and t_i is a term. If no rule applies, then there is no unifier; similarly, if any of the recursive calls to unify fail to produce a unifier, then there is no unifier. In the equations, v is an unbound variable, A is a term that is not a variable, f is an arbitrary symbol, and $\theta(A)$ is the result of applying substitution θ to term A. A leaf is equivalent to a function of no parameters, shown as $f()$ in the equations.

rocket. But that approach has a serious drawback. What would happen if, later, you tried to unify one of those two terms with term bird(feeder, Q). If you have already committed to X being rocket, then the later unification is impossible. But if you had seen ahead, and instead bound X and W each to feeder, the later unification would have succeeded.

That problem is overcome by performing unification in a most general way, allowing as many subsequent unifications to take place as possible. A unifier that is most general in this sense is called a *principal unifier*. Instead of replacing X and W each by a fixed value, a principal unifier chooses one of those variables to stand for both. Either X can be replaced by W or W can be replaced by X.

The definition of a principal unifier as a "most general" unifier can be made more precise. After performing the substitutions called for by a principal unifier θ, the effects of any other (possibly nonprincipal) unifier α can be accomplished by performing more substitutions (β) after the substitutions called for by θ. For example, $\alpha = \{X = \text{rocket}, W = \text{rocket}\}$ is a nonprincipal unifier for X and W. But it is equivalent to principal unifier $\theta = \{X = W\}$ followed by substitution $\beta = \{W = \text{rocket}\}$ since, for any term t, $\alpha(t) = \beta(\theta(t))$.

So the definition of a principal unifiers is, if A and B are two unifiable terms, then a unifier θ for A and B is *principal* if, for every unifier α for A and B, there exists a proper substitution β such that $\beta(\theta(A)) = \alpha(A)$. If two terms A and B can be unified, then there is always a principal unifier for them, and there is only one principal unifier, up to renaming of variables. (For example, a principal unifier might replace X by Y or Y by X, but the only difference is the choice of name, X or Y, for the variable.)

A principle unification algorithm

Figure 17.5 shows an equational definition of principal unification of terms. Notice that it uses the occur check, which is critical. Terms that can be unified without the occur check, but that cannot be unified with the occur check, have no unifier.

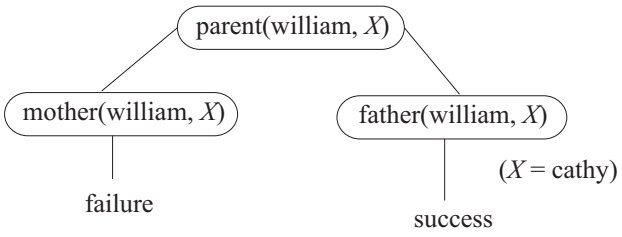


Figure 17.6: **Computation of goal parent(william, X), using the axioms from Figure 17.1.** Each step employs unifaction to make the head of the axiom equal to the current subgoal. Notice that unification not only specializes the axiom so that it will work for that subgoal, but also specializes the goal so that it can be proved by that axiom. Unification of goal father(william, X) and axiom father(william, cathy) causes unknown X to become bound to cathy, as is shown in the computation. The computation finds the value of X.

Practical unification

Binding an unknown requires replacing *every* occurrence of it, which is costly if the unknown is found in many places. An inference engine represents an unknown as a pointer, initially null to indicate an unbound unknown. To bind unknown X to tree t, the engine just makes X's pointer point to t. As long as the engine never makes a copy of an unknown, instead only pointing to it, there is only one change to make.

Any time it encounters an unknown with a nonnull pointer, the engine skips over the unknown by following the pointer. But remember that an unknown can be bound to another unknown, and that unknown might itself have become bound. The engine performs a loop, skipping over bound unknowns until it finds something that is not a bound unknown.

17.7. Goals involving unknowns

Imagine trying to prove goal parent(william, X), where X is an unknown, using the axioms of Figure 17.1. Computation of that goal is shown in Figure 17.6. During the process of the computation, unknown X becomes bound by unification. If you read backwards from the success to the root of the tree, and examine the bindings that are done, you can see what was proved; it was not really parent(william, X) that was proved, but the special case parent(william, cathy). Think of a goal containing unknown X not just as a yes/no question, but as a request that the engine find a value for X that makes the goal true, or to tell you that there does not exist any such value. In the example, you are asking for a child of william, and the engine tells you that cathy is a child of william.

In general, when a goal contains an unknown, the inference engine is asked to find a value for that unknown that makes proof of the goal possible, and unification is the tool that finds that value. So unification is not only responsible for specializing axioms to work for particular subgoals, but also for specializing subgoals so that they can be proved by particular axioms.

Notice the distinction between a goal and an axiom. An axiom is thought of as expressing a *property of a predicate*, namely, that the given fact is true for all values of the

variables. A goal, on the other hand, is thought of as stating a *property of the variables that the goal contains*, and the problem is to find values for the variables that have the desired property.

17.8. Auxiliary variables

Suppose that you want to define a new predicate grandparent, where grandparent(X, Y) is true when X is a grandparent of Y. You can express what it means to be a grandparent in (at least) two equivalent ways.

1. X is a grandparent of Y if there is some individual Z such that X is a parent of Z and Z is a parent of Y.
2. For any three individuals X, Y and Z, if it turns out that X is a parent of Z and Z is a parent of Y, then you can conclude that X is a grandparent of Y.

The second formulation avoids having to say that something exists, instead saying that a particular statement is true for all X, Y and Z. It leads to the following Horn clause defining predicate grandparent.

$$8.\ \text{grandparent}(X, Y) \Leftarrow \text{parent}(X, Z) \wedge \text{parent}(Z, Y).$$

If you think about using axiom (8) to prove a goal such as grandparent(bertrand, cathy), you see unknown Z popping up. The program does not say how to determine what Z should be, but that is all right, since it is the inference engine's responsibility to find things, and the engine should be able to find Z, without any help. The combined effects of backtracking and unification will find an appropriate Z automatically. Figure 17.7 shows computation of goal grandparent(bertrand, cathy).

To prove a list of goals, you start with the first one. Notice that replacing parent(bertrand, Z) by mother(bertrand, Z) does not affect the other goal, parent(Z, cathy), since that goal must still be proved later. But when Z is bound to marcy, it is replaced in the remaining goal, parent(Z, cathy), yielding goal parent(marcy, cathy).

Figure 17.8 shows computation of goal grandparent(X, marcy). Notice that both X = sally and Z = bertrand are found automatically. The program (or, if you prefer, the programmer) is not concerned with where they come from.

17.9. Computational rules

Renaming variables

Think, for the moment, of a procedural or functional language where you have a function $f(n)$. When you call this function the first time, you might choose $n = 2$. The next time, you could choose $n = 6$. Each time you use the function, you begin a new copy of the function, with a different variable n, and it is important to realize that there is more than one variable called n in the program.

The same kind of thing happens in a logic program. A given axiom might be used several times during a computation. Unifications that are done in one use of the axiom must not affect another use of the same axiom. If an axiom that includes variables is used more than once, a copy of the axiom should be made, with brand new variables that have not

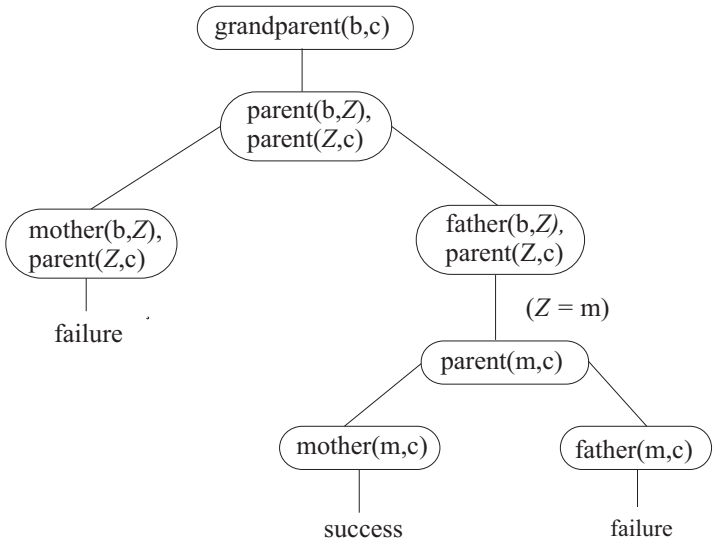


Figure 17.7: **Computation of goal grandparent(bertrand,cathy).** To reduce the size of the figure, each person's name is replaced with its first letter. For example, *b* stands for bertrand.

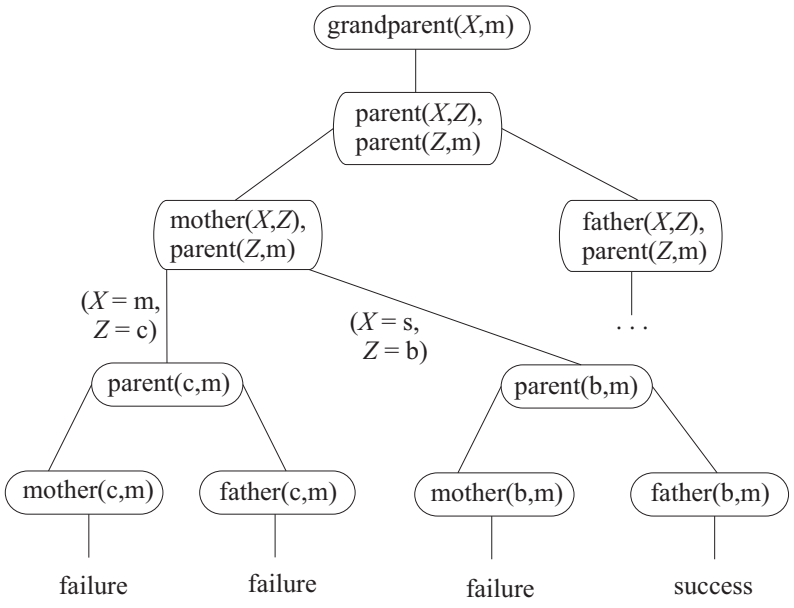


Figure 17.8: **Computation of goal grandparent(X,marcy).** Names are abbreviated to their first letters.

been used before. For example, if axiom (8) is used twice, the first time it might have the form (grandparent(X_1,Y_1) $\Leftarrow$ parent(X_1,Z_1), parent(Z_1,Y_1)), but the second time it could be (grandparent(X_2,Y_2) $\Leftarrow$ parent(X_2,Z_2), parent(Z_2,Y_2)). Renaming variables is especially important in logic programming because the variables that you create can remain unbound even after evaluation of a goal is finished. That is, a local variable can still exist, as an unbound unknown, even after the predicate that created it has been removed from the goal list.

Summary of computational rules

Here is a summary of the rules that you should keep in mind when building proof trees.

1. When a subgoal is replaced with a list of new subgoals, only that subgoal is replaced. Any other subgoals that do not participate are left in place, and must still be proved later.

2. When an unknown is bound, it must be replaced in *all* of the subgoals in the current list. For example, if the goal list is (mother(X, Z), parent(Z, marcy)), and you bind of Z to cathy in order to prove mother(X, Z), you need to bind Z to cathy in the second subgoal, parent(Z, marcy), as well.

3. Each time an axiom is used, you need to create a new copy of that axiom, with new variables, to keep different uses of an axiom from interfering with one another.

4. A successful proof is found when the goal list becomes empty. The bindings of the unknowns can be found by tracing up toward the root of the tree, reading off the bindings that were done by unification.

17.10. Recursion

Suppose you want ancestor(X, Y) to be true if X is an ancestor of Y. Two facts are evident.

9. ancestor(X, Y) $\Leftarrow$ parent(X, Y).
10. ancestor(X, Y) $\Leftarrow$ parent(X, Z), ancestor(Z, Y).

Figure 17.9 shows a proof of goal ancestor(bertrand, cathy).

17.11. Modes and information flow

If you think of a procedure in a procedural language, typically some of the parameters represent information coming into the procedure and some represent information coming out. For example, Ada procedure Sumdiff of Figure 6.7 has two in-parameters and two out-parameters. The procedure gets information from the in-parameters and stores information into the out-parameters.

Logic programming goals can have some parameters that are specific values and others that are unbound variables; the values represent information coming into the goal and the variables information that comes out (or that is left unknown). For example, in subgoal

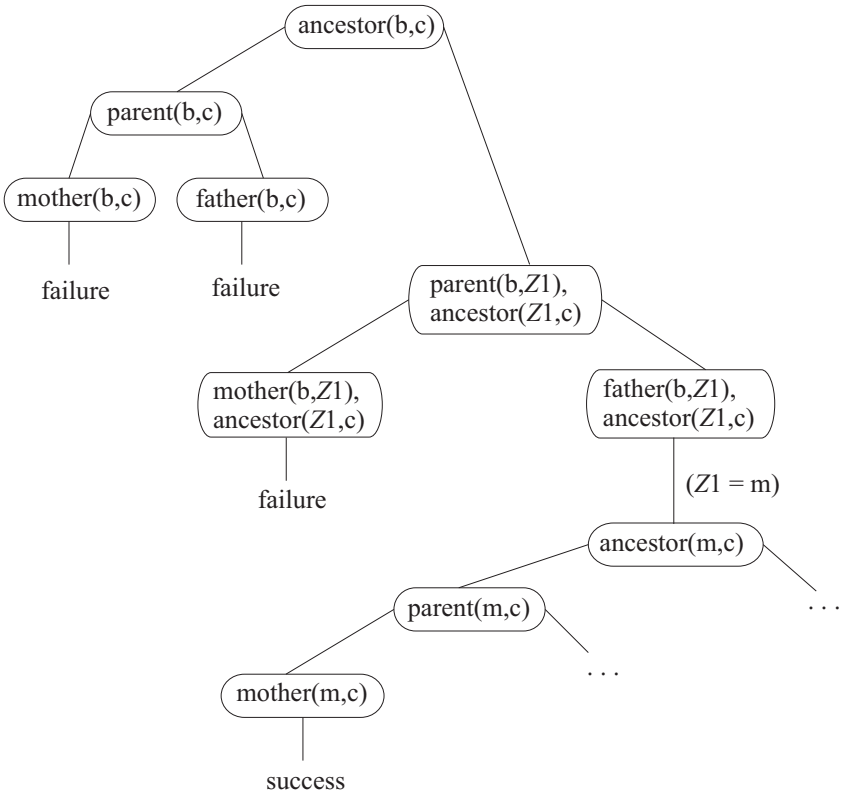


Figure 17.9: **Computation of goal ancestor(bertrand, cathy).** Unification performs quite a few variable bindings, but only the binding of Z_1 to m is shown explicitly.

mother(marcy, X), the first parameter is coming into the mother predicate and the second parameter is coming out.

The *mode* of a goal indicates which parameters are inputs and which are outputs. In goal grandparent(X, cathy), for example, the first parameter is an output and the second is an input, so the mode is grandparent(out, in), but goal grandparent(cathy, X) has mode grandparent(in, out). It is possible that there are no outputs, as in the pure test grandparent(bertrand,cathy), (mode grandparent(in, in)) or even that there are no inputs, as in grandparent(X,Y) (mode grandparent(out, out)). The grandparent predicate can be used in four modes: to compute a grandparent of an individual; to compute a grandchild of an individual; to test whether one individual is a grandparent of another; or to provide pairs (X,Y) where X is a grandparent of Y. In procedural programming, the mode is a property of a procedure, and a given procedure can only be used in one mode. In logic programming, the mode is a property of the goal, and a predicate can often be used in more than one mode.

The multimodal nature of predicates reduces the number of different predicates that need to be written. For example, the ancestor predicate can be used to compute descendants. Imagine that predicate sum(A, B, C) is defined so that sum(A, B, C) is true if $C = A + B$; that predicate can be used to add, of course, but it can also be used to subtract. If X and Y are known, then to compute $Z = X - Y$, simply try to prove sum(Z, Y, X), that is, $Z + Y = X$.

Not all predicates can be used in every possible mode. For example, the sum predicate might reasonably be asked to solve equation $Z = X + Y$ for any one of the variables, given the other two. But if more than one of the variables is unknown, then it becomes much more difficult to give a good definition of sum.

Typically, when you define a predicate, you also indicate the modes in which it will work. The sum predicate has four modes, sum(in, in, out), sum(in, out, in), sum(out, in, in) and sum(in, in, in). That is, at least two of its parameters must be known, and sum can determine the third. Out-parameters can always be replaced by in-parameters, since it is always acceptable to have a little more information than you really need. For example, since the sum predicate allows mode sum(in, in, out), it must also allow mode sum(in, in, in), indicating that it is just checking whether the sum is correct. The grandparent predicate can handle any mode at all, so it suffices to say that its mode is grandparent(out, out).

You can understand predicate modes by drawing diagrams that show how information flows into and out of predicates during computation of a predicate in a given mode. Figure 17.10 shows a diagram for a definition of the ancestor predicate when it is being used in mode ancestor(in, out). Drawing the information flow arrows requires that you think about the order in which the literals will be evaluated. Imagine computing a goal using axiom

$$\text{ancestor}(X, Y) \Leftarrow \text{parent}(X, Z), \text{ancestor}(Z, Y)$$

in mode is ancestor(in, out). You start out knowing X. The next subgoal that is evaluated is parent(X, Z); here, X is known, so parent needs to find Z. Then subgoal ancestor(Z, Y) knows Z (provided by parent), and needs to find Y, which is the value produced by goal ancestor(X, Y).

The ancestor predicate can be defined in several ways and the information flow can depend strongly on which definition is chosen. Figure 17.11 shows an alternative definition of ancestor and shows the information flow for the same mode, ancestor(in, out), as for the previous definition. Now, the parent predicate is used in mode parent(out, out), forcing it

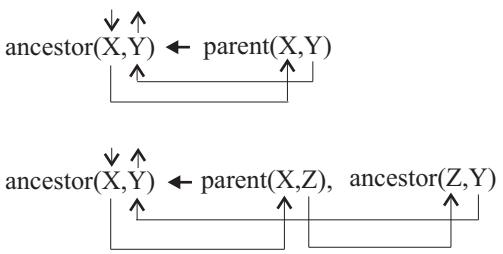


Figure 17.10: **A definition of predicate ancestor(X,Y), meaning that X is an ancestor of Y.** The arrows show the flow of information in the case where ancestor is used in mode ancestor(in, out). In the second axiom, the parent predicate is used in mode parent(in, out) and the recursive call to ancestor is also used in mode ancestor(in, out).

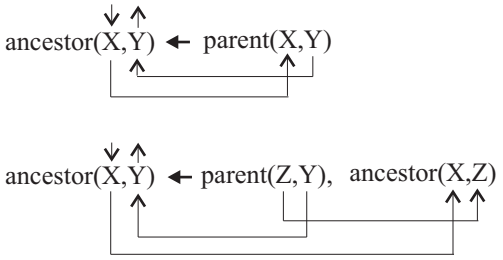


Figure 17.11: This definition of the ancestor predicate is different from the definition in Figure 17.10. The arrows show the information flow for the same mode as the arrows of Figure 17.10, ancestor(in, out). But in this case, the recursive call is not used in the same mode. Notice that, in the second axiom, the parent predicate mode is parent(out, out), which requires that all parenthood relations be tried. This definition is much less efficient than the former one *in this mode*.

to find all pairs (A, B) where A is a parent of B, and the recursive call to ancestor is used in mode ancestor(in, in), a pure test.

17.12. Understanding the meaning of a predicate

Every axiom in a logic program really has two meanings. In the *logical* interpretation, the axiom is taken as a statement of a fact about the predicate. But the axiom also makes a suggestion to the inference engine about how to use the axiom in proofs; it should replace the left-hand side (the head of a Horn clause) by the list of atoms on the right-hand side. So there is also a *procedural* interpretation of the axiom, where you think in terms of how the inference engine will use the axiom. If a predicate is intended to run in a restricted mode, you can find yourself tempted to think of the predicate and its axioms in procedural terms. For example, suppose that a list reversal predicate is intended to run in mode reverse(in, out). A procedural description of reverse is "reverse(X, Y) sets Y to the reversal of list X,"

since that is what the inference engine will do when it uses the reverse predicate in mode reverse(in, out).

But logic programming is not really intended to be procedural, and the semantics of a predicate should be logical, stated in terms of the values that make it true. When describing a predicate, think of its meaning in the mode where all parameters are inputs. For example, reverse(X, Y) is true when Y is the reversal of X (or, equivalently, when X is the reversal of Y). But the *mode of the description*, where all parameters are inputs, does not necessarily indicate the computational mode. The semantic description can be augmented by mode information, indicating how the predicate is intended to be used. A comment that the mode is reverse(in, out) suffices. Logic programming tries to separate logic from control, and you should maintain that separation in your descriptions.

17.13. Computing on trees

The predicates that we have written so far have only been concerned with simple data values, such a symbols and numbers. But logic programming generally works on trees, including the special case of lists. A good example is the append predicate, where append(A, B, C) is true if list C is A ++ B, and ++ is list concatenation. For example, append([2, 3], [5, 7], [2, 3, 5, 7]) is true. Axioms are as follows.

$$\begin{array}{l}\text{append}([\,],\ X,\ X).\\ \text{append}(H :: T,\ Y,\ H :: Z) \Leftarrow \text{append}(T,\ Y,\ Z).\end{array} \tag{17.13.1}$$

The first axiom states the obvious fact that $[\,] ++ X = X$. The second axiom states that $(H :: T) ++ Y = H :: (T ++ Y)$, or, equivalently, if $Z = T ++ Y$ then $H :: Z = (H :: T) ++ Y$.

It should come as no surprise that the definition of append can be used to concatenate lists. Goal append([2, 3], [4, 5], X) asks for X to be bound to [2, 3] ++ [4, 5], and the computation that leads to that is shown in Figure 17.12. Putting the bindings together, you see that $X = 2 :: Z1 = 2 :: 3 :: Z2 = 2 :: 3 :: [4, 5]$. That is, X = [2, 3, 4, 5].

But append can be used in a variety of modes. For example, goal append(X, [4, 5], [2, 3, 4, 5]) yields result X = [2, 3]. That goal is not using the append predicate to concatenate lists, but instead is using it to pull a list apart. Figure 17.13 shows how the inference engine processes it. As expected, it binds $X = 2 :: T1 = 2 :: 3 :: T2 = 2 :: 3 :: [\,] = [2, 3]$. Try computing goal append([2, 3], X, [2, 3, 4, 5]). You should find that X becomes bound to [4, 5]. Goal append(X, Y, [2, 3, 4, 5]) binds $X = [\,]$ and Y = [2, 3, 4, 5], and reaches a success. If backtracked into because that solution does not satisfy some later subgoal, it binds X = [2], Y = [3, 4, 5] for its second solution. In all, it produces five solutions, giving all pairs X and Y such that $X ++ Y$ = [2, 3, 4, 5]. You can even use goals where all of the parameters are unknown; the append predicate works in mode append(out, out, out).

When used in conjunction with the ability of the inference engine to find solutions via unification, the append predicate is remarkably powerful. For example, suppose that you want to write a predicate member(A, L) that is true just when A is a member of list L. You can write the member predicate using append.

$$\text{member}(A, L) \Leftarrow \text{append}(X,\ A :: Y,\ L). \tag{17.13.2}$$

Any time a variable occurs on the right-hand side of an axiom, but not on the left-hand side, you are asking the inference engine whether an appropriate value for that variable exists.

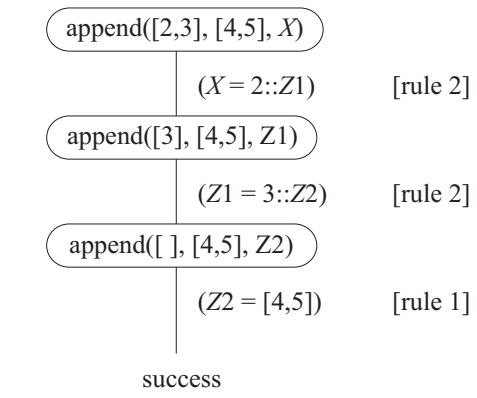


Figure 17.12: **Computation of goal append($[2,3]$, $[4,5]$, X) using Axioms 17.13.1.** It is critical to this computation that, each time an axiom is used, a new copy of it, with new variables, is created. The first time Rule 2 is used, its variable Z is called $Z1$, and the second time it is called $Z2$.

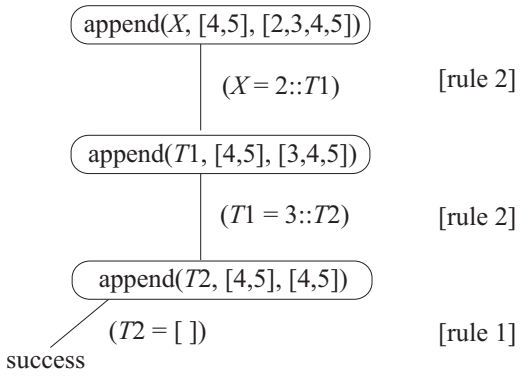


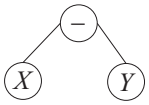
Figure 17.13: **Computation of goal append(X, $[4,5]$, $[2,3,4,5]$) using Axioms 17.13.1.**

Axiom 17.13.2 says that A is a member of list L if there exist lists X and Y such that $L = X{+}{+}(A :: Y)$. It is the responsibility of the inference engine to find X and Y, and unification will lead to them relatively quickly. Compare this with the definition of member using pattern matching from Section 16.3.2.

17.14. Difference lists and more general modes

Unification is so versatile that it is not necessary for every parameter to be entirely input or entirely output, even in a particular subgoal. For example, a parameter that is a list might contain some members that are known and some that are unknown. A predicate can examine the known members (making them inputs to the predicate), bind some unknown members (making them outputs) and just leave some of the unknown parts unknown, leaving them for other predicates to bind. Also, when a predicate binds an unknown, it might bind it to another partially known data structure.

A *difference list* is a tool that makes it easier to understand some computations on lists. If X and Y are lists where Y is a suffix of X, define the difference list $X - Y$ to be the list obtained from X by removing suffix Y from it. For example, [2, 4, 6, 8, 10] − [8, 10]) = [2, 4, 6]. If Y is not a suffix of X, then $X - Y$ is undefined. We are usually not interested in computing difference lists, but in using them in symbolic form as a way of understanding what a predicate does. Remember that $X - Y$ stands for a tree



so writing $X - Y$ is just a convenient way of combining variables X and Y.

For example, suppose that L is a (known) list of characters, representing a string, and that you want to get a decimal number from the front of L. What you want to find is the longest prefix N of L consisting entirely of decimal digits. But there might be additional useful information in L after the initial number, and you want to get that too, so you need to get the suffix S of L, after the numeric prefix. If L is ['2', '6', '1', ' ', '3', '2'], then the decimal number at the beginning of L is N = ['2', '6', '1'] (later to be converted to 261) and the rest of L is S = [' ', '3', '2']. Notice that $L - S = N$.

Figure 17.14 defines predicate extractNumber, where the idea is that extractNumber($L-S$, N) should be true just when S is a suffix of L and $L - S = N$ and N is the longest prefix of L that consists entirely of decimal digits. In terms of mode, extractNumber is intended to be used as extractNumber(in − out, out). That is, extractNumber($L - S$, N) takes in L and produces the prefix N and suffix S. For example, extractNumber(['2', '3', 'x', 'y'] − R, K) binds R = ['x', 'y'] and K = ['2', '3'].

There are other ways to write extractNumber. For example, it works just as well to have three parameters (L, S and N) rather than combining L and S into one parameter, but the idea of a difference list helps you see that you are breaking a list into a prefix and suffix, and makes the semantic intent clear.

17.14.1. Using difference lists for parsing

Chapter 3 introduces context-free grammars, and Figure 17.15 shows a small context-free grammar describing very simple English sentences. Suppose that you want to be able to

extractNumber([] − [], []).
extractNumber((H::T) − R, H::N) ⇐ digit(H), extractNumber(T − R, N).
extractNumber((H::T) − (H::T), []) ⇐ nondigit(H).

Figure 17.14: **Definition of predicate extractNumber.** The semantics is that extractNumber($L - S$, N) is true if S is a suffix of L and $L - S = N$ and N is the longest prefix of L that consists entirely of decimal digits. The intended mode is extractNumber(in − out, out). The first parameter is partially input and partially output. Predicate digit(C) is presumed to be true when character C is a digit, and nondigit(C) is true when C is not a digit.

⟨article⟩	::=	a | the
⟨propernoun⟩	::=	larry | curly | moe
⟨noun⟩	::=	parrot | kangaroo
⟨verb⟩	::=	meets | slaps
⟨nounphrase⟩	::=	⟨article⟩ ⟨noun⟩ | ⟨propernoun⟩
⟨verbphrase⟩	::=	⟨verb⟩ ⟨nounphrase⟩
⟨sentence⟩	::=	⟨nounphrase⟩ ⟨verbphrase⟩

Figure 17.15: **A grammar of simple sentences.**

recognize lists, such as [the, kangaroo, meets, larry], that are sentences according to this grammar. A natural plan is to write a predicate sentence(L) that is true just when L is a sentence. But suppose that, instead of taking the list directly, the sentence predicate takes a difference list. Define sentence($A - B$) to be true if $A - B$ is a sentence (and, of course, B is a suffix of A so that $A - B$ is meaningful). Then the sentence predicate does more than simply recognize a sentence; it can be used to extract a prefix of a list that is a sentence.

For each nonterminal N, you define a predicate $N(A - B)$ that is true just when list $A - B$ can be derived starting at nonterminal N. For example, in our small grammar, a sentence can be a noun phrase followed by a verb phrase, so, if $L - R$ is a sentence, then $L = X +\!\!+ Y +\!\!+ R$ where X is a nounphrase and Y is a verbphrase. But we need to express the noun phrase X and verb phrase Y as difference lists, since that is the form that our predicates expect. Since $L = X +\!\!+ Y +\!\!+ R$, list X can be defined as $X = L - (Y +\!\!+ R)$. In logic programming, you generally need to give things names, so define $A = Y +\!\!+ R$. Then the noun phrase X is $L - A$, and the verb phrase Y is $A - R$ (since $Y = (Y +\!\!+ R) - R$). Expressing that X is a noun phrase and Y is a verb phrase yields the following axiom.

$$\text{sentence}(L - R) \Leftarrow \text{nounphrase}(L - A), \text{verbphrase}(A - R) \tag{17.14.1}$$

Figure 17.16 shows axioms for the entire grammar. Goal sentence([larry, slaps, curly] − []) should succeed.

article((a::R) − R).
article((the::R) − R).
propernoun((larry::R) − R).
propernoun((curly::R) − R).
propernoun((moe::R) − R).
noun((parrot::R) − R).
noun((kangaroo::R) − R).
verb((meets::R) − R).
verb((slaps::R) − R).
nounphrase($L - R$) ⇐ article($L - A$), noun($A - R$).
verbphrase($L - R$) ⇐ verb($L - A$), nounphrase($A - R$).
sentence($L - R$) ⇐ nounphrase($L - A$), verbphrase($A - R$).

Figure 17.16: **Implementation of the grammar of Figure 17.15.** Each predicate describes what it means for a difference list to be derivable from a given nonterminal. Upper case letters are variables and words that start with lower case letters are (constant) symbols. A grammar written in this style is called a *definite clause grammar*.

17.15. Positive thinking

It is possible to define the member predicate directly, without using append. Here it is.

$$\begin{array}{l} \text{member}(A, A :: X) \\ \text{member}(A, B :: X) \quad \Leftarrow \quad \text{member}(A, X) \end{array} \tag{17.15.1}$$

Compare that with an equational definition of member.

member(a, [ ]) `= false`
member(a, $a :: x$) `= true`
member(a, $b :: x$) `= member(`a`, `x`) when` $b \neq a$

Notice that the two cases of the logic programming definition correspond roughly to the last two cases of the equational definition. (Why is it possible to drop the proviso that $b \neq a$?) There is no case that corresponds to the equation member(a, [ ]) = false. In logic programming, you say what is true. Anything that does not follow logically from what you have said is presumed to be false, a presumption that is captured in the *closed world hypothesis*.

> **Closed World Hypothesis.** The program says everything that is relevant. Anything that is not said, and that does not follow logically from what was said, is false.

As another example, consider the problem of telling whether a list is a palindrome, a list that is its own reversal. One approach is as follows. Notice that there is no negative information. Only the positive cases are stated.

palindrome([ ]).
palindrome([A]).
palindrome(A::X) ⇐ append(Y, [A], X), palindrome(Y).

The third fact states that, if Y is palindrome and $X = Y +\!\!+ [A]$, then $A :: X$ is a palindrome. That is, $A :: (Y +\!\!+ [A])$ is a palindrome.

17.16. Computing functions

In imperative or equational programming, list concatenation is normally provided as a function (such as operator ++) taking two parameters. In logic programming, concatenation is provided as a predicate (append) having three parameters, where an extra parameter is added for the result. You generally add an extra parameter any time you need to compute a function via a predicate. For example, to compute the N-th member of list L, you need a third parameter to hold the result, so you define predicate nth(N, L, R), which is true just when R is the N-th member of L. An exception is a function that produces a boolean result, since predicates naturally produce true or false already. For example, there is no need to add an extra parameter to the member predicate.

In a style that uses expressions, you set $a = x \mathbin{++} y \mathbin{++} z$ as follows.

```
Let a = x ++ (y ++ z).
```

The intermediate result $y \mathbin{++} z$ is implicitly passed to the next use of ++. But using predicates, you need to introduce a variable to stand for the intermediate result, and pass it from one predicate to the next using the idea of information flow. For example,

```
append(Y, Z, R),
append(X, R, A).
```

states that $R = Y \mathbin{++} Z$ and $A = X \mathbin{++} R$.

17.17. Exercises

17.1. What is the definition of a Horn clause? What is the head of a Horn clause?

17.2. Does logic programming, as described in this chapter, allow an axiom to be any formula at all, or is there a restriction on kinds of formulas that can be used? Explain.

17.3. When an unknown X occurs in an *axiom*, is the engine being told that the axiom holds for all values of X, or for some value of X?

17.4. When an unknown X occurs in a *goal*, is the engine being asked whether the goal holds for all values of X, or for some value of X?

17.5. Show each of the following as a tree diagram.

(a) bear(black)

(b) fruit(mango, citrus(orange, tangerine))

(c) $f(X, g(Y))$

17.6. Unification is a form of pattern matching. Which of the following is *not* a characteristic of unification?

(a) Unification never changes the binding of a bound variable.

(b) Unification is symmetric; unifying A with B has exactly the same effect as unifying B with A.

(c) Unification performs arithmetic, so binding X to $5 + 6$ will bind X to 11.

(d) Unification can bind unbound variables.

17.7. Show tree diagrams for each of the following pairs, and show how unknowns become bound when the pair is unified, by showing the pointers. There might be more than one way to do some of them, since to unify X with Y, you can either replace X by Y or Y by X.

(a) $f(X, Y)$ and f(apple, orange)

(b) $f(X, Y)$ and f(apple, X)

(c) $f(g(X, Y), Z)$ and $f(Z, W)$

(d) $r(g(X), Y, h(Z))$ and $r(Y, W, X)$

17.8. Give a principle unifier for each pair of terms in problem 17.7.

17.9. What happens if you unify terms $f(X, Y)$ and $f(g(Y), X)$ using naive unification? Show the result in a tree diagram, with pointers. Can those terms be unified using full unification?

17.10. Give a principal unifier and two different examples of non-principal unifiers of terms $f(X, Y)$ and $f(Z, W)$. Then show how to accomplish the same effect as each of those non-principal unifiers in two steps, first by using a principal unifier, then by performing another simple substitution.

17.11. Why do you think an unknown is only allowed as a leaf of a tree? (Remember that an unknown can be bound to a tree.)

17.12. Using Horn clauses, write a definition of predicate sameParent(X, Y), which is true if individuals X and Y have at least one parent in common. (This indicates that X and Y are either (half) siblings or are the same individual.)

17.13. Using Horn clauses, write a definition of predicate greatGrandparent(X, Y), which is true if X is Y's great grandparent.

17.14. Using Horn clauses, write a definition of predicate cousinOrCloser(X, Y), which is true if X and Y are either cousins or siblings or the same individual. (Cousins share a common grandparent.)

17.15. Using Axioms (1–8), show a proof tree for goal grandparent(jordan, X). Where does the first success occur?

17.16. Using Axioms (1-10), show a proof tree for goal ancestor(A, cathy), showing all successes.

17.17. Can the ancestor predicate be used to find descendants? Using Axioms (1–10), show a proof tree for goal ancestor(sally, X), and show all of the branches that lead to success. Label each success by the value of X that it finds.

17.18. Using Axioms (1–8), show a proof tree for goal parent(A, B), showing all successes, not just the first one. Does it make sense to ask the engine to provide more than one value?

17.19. Using Axioms (1–8), show a proof tree for goal grandparent(X, Y), up to the point of the first success. What are X and Y? What does goal grandparent(X, Y) ask for?

17.20. In what modes can predicate member defined by Axiom (17.13.2) be used?

17.21. Draw an information flow diagram for the definition of append (Axioms 17.13.1) in mode append(in, in, out).

17.22. Draw an information flow diagram for the definition of append in mode append(in, out, in). Is the recursive call done in the same mode?

17.23. Show the information flow for the ancestor predicate in mode ancestor(out, in)

(a) using the definition of ancestor in Figure 17.10.

(b) using the definition of ancestor in Figure 17.11.

17.24. Show the information flow for Axiom (17.14.1) when used in mode sentence(in−out, out).

17.25. Write axioms for predicate even(L), which is true if list L has an even number of members. In mode even(out), it should bind its parameter to a list of unbound variables, first of length 0, then 2, then 4, etc.

17.26. Write axioms for predicate samelength(A, B), which is true when A and B are lists of the same length. Make it usable in mode samelength(out, out). If both parameters are unknown, samelength should bind each of them to a list of zero or more unknowns, of the same length. (Try length 0, then one, then two, etc.)

17.27. Write axioms for predicate allsame(L), which is true just when all members of list L are the same. For example, allsame($[5, 5, 5]$) is true. By definition, allsame($[\,]$) and allsame($[x]$) are true.

17.28. Write axioms for predicate prefix(X, Y), which is true if list X is a prefix of list Y. For example, prefix([a, b], [a, b, c, d]) is true. The empty list is a prefix of every list, and every list is a prefix of itself.

17.29. Write axioms for predicate suffix(X, Y), which is true if list X is a suffix of list Y. For example, suffix([c, d], [a, b, c, d]) is true. A list is a suffix of itself. For this definition, do not use recursion, but use the append predicate to do the work for you.

17.30. Convert the definition of naive reverse (Equation 9.9.1) to a logic programming style. Use append to perform concatenations.

17.31. Naive reverse is inefficient. Write a more efficient implementation of reverse, as a predicate, following the plan of Equations 9.9.2, 9.9.3 and 9.9.4.

17.32. Write a goal, using the append predicate, to bind X to the last element of list L, assuming that L is nonempty. Do not write any new predicate definitions. Just write a goal.

17.33. Using the goal from the preceding exercise, write a definition of predicate last(L, X) that is true if L is a nonempty list whose last member is X. Do not use recursion, just use the append predicate to do the job.

17.34. Write axioms for predicate twice(X, L), which is true if L has the form X ++ X. For example, twice([3, 5], [3, 5, 3, 5]) is true. It should be usable in mode twice(out, out).

17.35. Write a definition of predicate perm(A, B) that is true if list B is a permutation of list A. Determine the modes in which your definition can be used.

17.36. Show the computation tree for goal list (member(X, [3, 4, 5]), member(X, [4])), where X is an unbound variable, up to the point where the first success is found, using Axioms 17.15.1 as the definition of member.

17.37. Suppose that list membership is defined by Axioms 17.15.1. Can member be used in mode member(in, out) to produce a list that has a given value as a member? Show a computation of member(3, L) to illustrate. What are the first to results for L?

17.38. Suppose that the two axioms of the preceding exercise are written in the opposite order,

$$\text{member}(A, B :: X) \Leftarrow \text{member}(A, X).$$
$$\text{member}(A, A :: X).$$

Can this definition of member be used in mode member(in, out)? Try computation of member(3, L). Does the order in which axioms are written matter?

17.39. Using the style described in Section 17.13, write a parser for expressions using grammar (3.2.8) in Chapter 3. For simplicity, presume that a number is a single digit.

17.18. Bibliographic notes

Colmerauer and Rousel [29] discusses the history of the development of logic programming and the Prolog programming language. See the next chapter for more notes.

Hogger [52], Clocksin and Mellish [28] and Kowalski [60] explain principles and uses of logic programming.

Chapter 18

Prolog

18.1. Introduction to Prolog

Prolog is a typeless programming language, or really a family of related programming languages, that is dedicated to logic programming. This chapter looks primarily at ISO Standard Prolog, simply called Prolog here. ISO Prolog has some differences from typical other dialects, and this chapter mentions a few of them.

Basic syntax

In Prolog, each axiom or goal must be followed by a period. The right-to-left implication $\Leftarrow$ is written `:-`. For example, an axiom defining the grandparent predicate is written as follows in Prolog.

```
grandparent(X,Y) :- parent(X,Z), parent(Z,Y).
```

Prolog is mostly free-form, so an axiom can extend over several lines, with a period used to indicate the end of the axiom. In ISO Prolog, you write a comment inside /*...*/. Many Prolog dialects use a comment form that begins with % and continues to the end of the line.

In ISO Prolog, symbols can be written in three forms: (1) a name, such as chihuahua, that starts with a lower case letter and can contain letters, digits and the underscore character; (2) a sequence of special characters, such as + and +@$, consisting of one or more special characters in string "`+-*/\^<>:.?@#$&`"; and (3) any sequence of characters enclosed in single quotes, such as 'the man in the moon'. Type (3) can only be constants or leaves of trees; types (1) and (2) can occur anywhere in a tree. Predicate names are required to have the same form as type (1) symbols. Different dialects of Prolog have different requirements on symbols. It is a good idea to restrict yourself to names that contain only letters, as well as a few short symbolic names such as + and −.

Prolog is case-sensitive. Unknowns are called *variables*, and variable names start with an upper case letter or underscore, and can contain letters, digits and the underscore character. So, while kangaroo is a symbol or predicate or function, Kangaroo is a variable. A single underscore is an anonymous, or don't-care, variable; each underscore is a different variable.

```
append([], X, X).
append([H|T], Y, [H|Z]) :- append(T, Y, Z).

member(A, [A|_]).
member(A, [_|X]) :- member(A, X).
```

Figure 18.1: **Axioms for the append and member predicates in Prolog notation.** Recall that append(X, Y, Z) is true if Z is list X followed by list Y, and member(X, L) is true if X is a member of list L. An underscore is an anonymous variable, and is commonly used for a variable that occurs only once in an axiom.

You should be able to put a space between any two tokens. Some Prolog dialects are picky about spaces in certain places. In particular, a space between a predicate name and the left parenthesis just after that name confuses some Prolog compilers.

Functors and operators

Functions do not perform computation in Prolog. They are only used to describe trees. For example, $g(X, a)$ indicates a tree with root labeled g, and two subtrees X and a. To emphasize the distinction between the Prolog view of functions, as tags in trees, and functions in most programming languages, which perform computations, Prolog uses the term *functor*. So $g(X, a)$ is a tree with functor g. Prolog allows you to use binary operators such as $+$, $-$, $*$ and $/$, but $X + Y$ has the same meaning as $+(X, Y)$, which describes a tree with root labeled $+$ and two subtrees, X and Y.

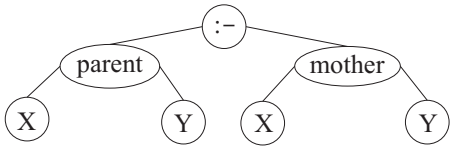
Lists

The empty list is written [ ]. The Cinnameg :: operator is written as a dot in Prolog, and you can write $.(X, Y)$ for the list whose head is X and whose tail is Y. Lists are special cases of trees, and $.(X, Y)$ is just a tree whose root is labeled by a dot, with left subtree X and right subtree Y.

Prolog provides a notation, $[H|T]$, that has the same meaning as $.(H, T)$, and that is the notation that we will use. (The bar notation is more universal among different dialects than the dot notation, so it is a good choice.) You must use brackets with this notation; Prolog does not allow you to write $H|T$ without brackets around it. Figure 18.1 shows definitions of the append and member testing predicates. An extension of the list notation is [A, B, C | R], which is the list that begins [A, B, $C, \ldots$], with the rest ($\ldots$) being R. So [A, B, C | R] stands for $.(A, .(B, .(C, R)))$. You can list any number of members before the bar, but just one (the rest of the list) after the bar.

18.2. Prolog implementations

Prolog implementations come in three forms. One is an interpreter that works on a direct description of the axioms. The axioms are read and converted into a tree structure, and the interpreter works on the trees. For example, axiom parent(X,Y) `:-` mother(X,Y) is represented as follows by the interpreter.



A second kind of implementation translates the axioms into instructions of an abstract machine, and uses an interpreter to run the abstract machine. So instead of seeing an axiom, such as append([ ], X, X) for the append predicate, the interpreter is given instructions that, in conceptual form, are something like the code shown in Figure 18.2. Basic operations such as unification and operations on the run-time stack and the control stack, as well as handling of the trail, are built into the implementation of the abstract machine.

A third kind of implementation is a compiler that compiles to native (machine) code, or to some language such as C that can be compiled into native code. We will restrict attention here to implementations that employ interpreters.

Interaction with the Prolog interpreter

Prolog systems typically interact with the user, asking for goals. A goal begins with `?-` and consists of a list of comma-separated atomic formulas, ended by a period. So, to ask whether list [a, b, c] contains b, you type goal

```
?- member(b, [a,b,c]).
```

The system responds `yes`. Notice that a, b and c are symbols, not variables, since they begin with lower case letters.

When a goal contains one or more variables, the system responds with the values of the variables that make the goal true. A semicolon typed by the user asks for yet another solution. It forces backtracking by causing an artificial failure. An interaction with a Prolog system might go as follows. The goal and the semicolons are typed by the user. The values of X are written by the interpreter.

```
?- member(X, [a,b,c]).
X = a;
X = b;
X = c;
no
```

You will usually want to load axioms into the interpreter from a file. Pseudo-goal

```
consult('myfile.pl')
```

loads the axioms found in file myfile.pl. Many Prolog implementations allow a goal to be a list of symbols. All of the files named in the list are consulted. Also, some versions of Prolog require file names to be strings enclosed in double quotes rather than symbols.

18.3. Unification

Chapter 17 discusses unification, which involves a test called the *occur check*. Most dialects of Prolog omit the occur check because it is expensive to perform, a consequence being that

```
PUSH-MARK-FRAME
PUSH-BACKTRACK-FRAME (pointing to Label 1)
V1 = new variable (X)
UNIFY param1, []
UNIFY param2, V1
UNIFY param3, V1
POP-MARK-FRAME
RETURN
LABEL 1
V1 = new variable (H)
V2 = new variable (T)
V3 = new variable (Y)
V4 = new variable (Z)
V5 = new variable (T)
V6 = V1.V2
V7 = V1.V4
UNIFY param1, V6
UNIFY param2, V3
UNIFY param3, V7
CALL append(V5,V3,V4)
POP-MARK-FRAME
RETURN
```

Figure 18.2: **A translation of the definition of append into a generic abstract machine language.** The backtrack frame will start the second axiom if the first fails, due to failed unifications. The marker frame is used to limit how far back a cut goes. You can see that the call to append in the second axiom is tail-recursive, since, ignoring the marker frame pop, it is immediately followed by RETURN, so an efficient call instruction that will only work for tail-recursive calls could be used instead of a normal call. The last three instructions (starting at the call) can be replaced by

```
POP-MARK-FRAME
TAIL-CALL append(V5,V3,V4)
```

The POP-MARK-FRAME needs to be done before the tail-call because TAIL-CALL will not return; it destroys the frame for this predicate, so it has nothing to return to.

it is possible to create cyclic data structures, representing infinite trees. ISO Prolog requires the occur check to be performed, and disallows cyclic structures.

Goal $X = Y$ is performed by unifying X and Y. You could write your own predicate eq(X, Y) to perform the same action, by writing a single axiom, `eq(X,X)`, and writing eq(A, B) instead of $A = B$.

18.4. Disjunction

On the right-hand side of an axiom, a comma indicates conjunction; to say that you have proved the right-hand side, you need to prove all of the goals that are listed. Prolog also provides a disjunction operator, a semicolon, indicating an *or* of parts. For example,

```
parent(X, Y) :- mother(X, Y) ; father(X, Y)
```

says that parent(X, Y) is true when either mother(X, Y) is true or father(X, Y) is true. Prolog implementations deal with semicolons by translating to more than one axiom. The axiom above is equivalent to

```
parent(X,Y) :- mother(X,Y)
parent(X,Y) :- father(X,Y)
```

18.5. Definite clause grammars

Section 17.14.1 describes definite clause grammars, which use logic programming with difference lists to parse sequences of words according to a context-free grammar. Some Prolog dialects have a special notation that makes writing definite clause grammars easy and convenient. ISO Prolog provides notation

```
sentence --> nounphrase, verbphrase.
```

which stands for

```
sentence(A, R) :- nounphrase(A, B), verbphrase(B, R).
```

No explicit difference list is built, but goal sentence(A, R) is thought of as meaning that $A - R$ is a sentence. There can be any number of parts on the right-hand side of `-->`; they are chained together, letting each extract a prefix of a list and pass the remainder to the next predicate. There is also a notation for rules that produce fixed words. Notation

```
noun --> [horse].
```

stands for

```
noun([horse|X], X).
```

and fixed parts (in brackets) can be mixed with nonterminals in rules, as shown in Figure 18.3.

Sometimes you need to add more parameters to the parsing predicates. For example, rather than merely checking whether a list is a sentence, you might like to produce a parse tree of a sentence. Prolog's grammar notation allows you to add parameters, which are just added to the parsing (difference list) parameters. For example, rule

```
expression(e(T))     --> term(T).
expression(e(T,+,E)) --> term(T), [+], expression(E).
term(t(N))           --> num(N).
term(t(N,*,T))       --> num(N), [*], term(T).
num(N)               --> [N], {number(N)}.
```

Figure 18.3: **A parser for simple expressions.** Goal expression(T, L, R) succeeds if difference list $L - R$ is an expression according to this grammar and T is a parse tree for expression $L - R$. Parameters L and R are not written explicitly. They are added when these grammar rules are translated into axioms. Goal expression(T,L,R) normally takes L in and produces T and R. For example, goal expression(T, [2,+,3,*,4], [ ]) succeeds, binding T to $e(t(2), +, e(t(3, *, t(4))))$. Standard predicate number(N) succeeds when N is a number, and {number(N)} is just a test ensuring that N is a number.

```
sentence(sent(NP,VP)) --> nounphrase(NP), verbphrase(VP).
```

is translated to

```
sentence(sent(NP,VP), A, R) :- nounphrase(NP, A, B),
                               verbphrase(VP, B, R).
```

Additionally, you can add tests that are not part of the grammar rule by enclosing parts in braces; those parts are copied directly into the translation, without change, and without affecting the difference-list parameters. Figure 18.3 shows a Prolog parser for expressions that employs all of these features.

18.6. Reading and writing

In ISO Prolog, a character is just a symbol whose name has length 1. For example, symbol x is thought of also as the character x. Some dialects just use character codes (integers), and dispense with the notion of a character.

Goal read(X) reads a Prolog term from the standard input, and unifies X with that term. To read just one character, use get_char(X). (In some dialects, goal get0(X) binds X to the next character code, an integer.)

You can write a term X to the standard output using write(X). Goal put_char(X) writes character X. (Some Prolog dialects use put(N) instead of put_char(X), where N is an integer code.) Goal nl ends a line on the standard output.

18.7. Arithmetic in Prolog

Arithmetic creates a problem for Prolog. All computation is done by predicates, not by functors. Goal `2 + 3 = 3 + 2` fails, since 2+3 and 3+2 are different trees. Prolog deals with arithmetic by introducing a special operator, **is**, where goal A **is** B is performed by evaluating B using the usual rules of arithmetic and then unifying A with the result. For example, goal list

```
X is 2 + 3, Y is 3 + 2, X = Y.
```

```
factorial(0,1).
factorial(X,Y) :-
  X > 0,
  N is X-1,
  factorial(N,M),
  Y is X*M.
```

Figure 18.4: **Definition of factorial in Prolog.**

succeeds, since X and Y are each bound to 5. Operators that **is** understands include $+$, $-$, $*$, / (real division), // (integer division) and mod (integer remainder). Be careful; goal (2 + 3 is 3 + 2) still fails, since only the right-hand side, $3+2$, is evaluated by the **is** operator.

Goal (K is X + Y) causes an error if either of X or Y is unknown, since arithmetic cannot be performed on unknown numbers. So you need to be careful about mode when using the **is** operator. For example, goal ($X = 3$, $Y = 4$, 7 is $X+Y$) succeeds, but evaluating goal (7 is $X+Y$, $X = 3$, $Y = 4$) causes an error, since X and Y are not known at the point where arithmetic is requested.

You can compare two (known) numbers using operators =:= (equality), =\= (inequality), <, >, =< and >=. (Notice that Prolog uses =<, not <= for a less-than-or-equal test.) For example, goal $3 < 6$ succeeds.

Example

Figure 18.4 shows a definition of a factorial predicate in Prolog, where factorial(N, Y) is true if $Y = N!$. Goal factorial(3, N) binds $N = 6$. But goal factorial(M, 6) yields an error, since the $>$ operation is used with an unbound variable. Prolog is obviously not well-suited to performing arithmetic calculations. It is designed for symbolic computing.

Notice that the second axiom for factorial explicitly states that X must be greater than 0. Is that really required? You know that the axioms are tried in the order written, so you might argue that if the first parameter is 0, then the first axiom will be taken. But in a logic program, *all* of the axioms are tried, by backtracking, and more than one can succeed. For example, what if you compute goal list (factorial(0, N), $N = 12$)? Clearly, that goal must fail. Initially, factorial binds $N = 1$, which leads unification $N = 12$ to fail. But then factorial is backtracked into, and the second factorial axiom is tried, asking factorial whether it wants to produce yet another answer. Without the test that $X > 0$, that leads to computation of the factorial of -1, and an infinite recursion.

Using jailbird notation

If you only need small nonnegative integers, you can use a representation that exploits Prolog's data structure capabilities to get more versatile operations on numbers. Instead of using the built-in numbers, use a list of length n to stand for the number n. For example, list [i, i, i] stands for 3. The advantage is that you can perform arithmetic and comparisons on unknown numbers. To add two numbers, just concatenate them, using the append predicate. To subtract, just use the append predicate in a different mode. For example, saying that $X = Y - Z$ is equivalent to saying that $Y = X + Z$, which is expressed as append(X, Z, Y). That will only succeed when the difference $Y - Z$ is nonnegative, since lists have

```
take([], _, []).
take([i|N], [H|T], [H|R]) :- take(N, T, R).
```

Figure 18.5: **Definition of predicate take using jailbird notation.** The meaning is that take(N, X, Y) is true if list Y is the length N prefix of X. Number N is represented as a list for versatility, allowing take to be used in different modes. If N is larger than the length of X, then take(N, X, Y) fails. It is no surprise that take([i, i], [a, b, c, d], L) binds L to [a, b]. But this version of take will also handle goal take(N, [a, b, c, d], [a, b]), where N is unknown; it binds N to [i, i].

```
factorial(0,Y) :- !, Y = 1.
factorial(X,Y) :-
  N is X-1,
  factorial(N,M),
  Y is X*M.
```

Figure 18.6: **An alternative Prolog definition of the factorial predicate.** Instead of testing whether $X > 0$ in the second axiom, the first axiom cuts away the branch that would have tried the second one, ensuring that the second axiom will not be tried for $X = 0$. (This definition assumes that X is not negative.)

nonnegative length. You can express $X + 1$ as [i $|X$]. To compare numbers, compare the lengths of lists. For example, define ge(X, Y) to be true when X is at least as long as Y, as follows.

```
ge(_,[]).
ge([i|A], [i|B]) :- ge(A,B).
```

Figure 18.5 shows an implementation of the take function, which takes a prefix having a given length from a list, as a predicate. Because numbers are represented in jailbird notation, the take predicate can be used in several modes. Had usual arithmetic been used, the mode would have been restricted.

18.8. Pruning the proof tree using cuts

The backtrack control mechanism is highly demanding of memory, and practical Prolog programs usually need to employ commits (Section 16.4.3) to destroy control frames. Prolog calls a commit a *cut*, and uses an exclamation point to indicate a cut; the cut removes all backtrack frames back to one indicated by the topmost marker frame. As Figure 18.2 shows, a marker frame is pushed each time evaluation of a predicate begins.

The factorial predicate defined in Figure 18.4 requires an explicit test $X > 0$ in its second axiom to prevent that axiom from being used when only the first axiom should apply. Alternatively, you can omit the test by using a cut to remove the branch that would try the second axiom, as shown in Figure 18.6.

If-then-else

Cuts allow you to simulate an if-then-else construct. To make $P(X)$ simulate **if** $A(X)$ **then** $B(X)$ **else** $C(X)$, define $P(X)$ as follows.

```
P(X) :- A(X), !, B(X).
P(X) :- C(X).
```

When A(X) succeeds, the cut removes the branch that would have done C(X).

Cuts limit mode

The membership testing predicate can also, at least apparently, benefit from a cut. Once a value x is found in a list, there appears to be no point in looking for another copy of it, since you only want to know whether there are any occurrences of x. Axioms

```
member(A, [A|_]) :- !.
member(A, [_|X]) :- member(A, X).
```

accomplish that. That works well in mode member(in, in). But what if it is used to generate members of a list, in mode member(out, in), as in goal `member(`X`, [1,2,3,4]), `X `= 2`. If the cut were not present, that goal would succeed and bind $X = 2$. But the presence of the cut means that, after binding X to 1, the second member axiom will not be tried. So the possibility that $X = 2$ will never be tried, and the goal fails.

As a rule, using a cut limits the mode in which a predicate can be used, so be cautious. When justifying the use of a cut, always consider the modes in which you want your predicate to work.

18.9. Negation

Suppose that you want to write a predicate sibling(X,Y) that is true when X and Y are (half) siblings, with the intended mode sibling(in, in). A first attempt is as follows.

```
sibling(X,Y) :- parent(Z,X), parent(Z,Y).
```

Unfortunately, according to this definition, everybody who has a parent is his or her own sibling, since there is nothing to prevent X and Y from being the same individual. Somehow, you need to say that $X \neq Y$. Unification can be used to state that two variables are equal, but not to state that they are not equal. Remember that, when a Horn clause is written in the usual form, as an implication, all literals are positive, and there is no provision for saying that anything is false. Let's try adding a special case to handle equality, explicitly saying that sibling(X, X) is false. Standard predicate **fail** takes no arguments and always fails.

```
sibling(X,X) :- fail.
sibling(X,Y) :- parent(Z,X), parent(Z,Y).
```

But that does not do the job; the first axiom always fails, and the second axiom is tried anyway. A cut will prevent the second axiom from being tried.

```
sibling(X,X) :- !, fail.
sibling(X,Y) :- parent(Z,X), parent(Z,Y).
```

The cut effectively implements an inequality test.

Negation as failure

In general, if you have any predicate p(x), you can define another predicate notp(x), where notp(x) is true just when p(x) is false, as follows.

```
notp(X) :- p(X), !, fail.
notp(X).
```

If p(X) succeeds then notp(X) fails by the first rule. It also cuts away backtracking so that the second rule cannot make notp(X) succeed. But if p(X) fails, then the cut in the first rule will not be done, and the second rule will be backtracked into, causing notp(X) to succeed. This approach to simulating negation is called *negation as failure*.

ISO Prolog provides negation as failure as a unary operator, \+; goal \+G succeeds just when an attempt to prove goal G fails. So sibling(X, Y) can be defined as follows.

```
sibling(X,Y) :- parent(Z,X), parent(Z,Y), \+(X = Y).
```

Some other dialects of Prolog offer the same thing as a pseudo-predicate not(A), and the sibling predicate in those dialects has the following form.

```
sibling(X,Y) :- parent(Z,X), parent(Z,Y), not(X = Y).
```

Negation and mode

You should be wary of negation as failure. It is implemented using a cut, and cuts usually limit the mode in which a predicate works. Consider the following goal.

```
?- \+(X = 2).
```

This goal contains a variable X, so it appears to ask for some value X such that $X \neq 2$. Of course, there are many such values X. But the Prolog system will not find any of them. First, it tries to prove $X = 2$. That succeeds, binding X to 2. But because goal $X = 2$ succeeds, goal \+($X = 2$) fails! The \+ operator is only a correct implementation of the concept of negation if no variables (that exist when \+ starts) are bound during its computation. Goal list ($X = 4$, \+($X = 2$)) succeeds, with $X = 4$, since X is already bound when the negated goal is tried. But goal list (\+($X = 2$), $X = 4$) fails, because X is unbound when the negated goal is tried.

If you think about what negation as failure does when variables become bound, you can see that it is performing a kind of nonexistence test. If X is unbound when goal \+($X = 2$) is tried, then this goal is asking whether it is true that there does not exist an X such that $X = 2$. Of course, that is false, since there does exist an X such that $X = 2$, so goal \+($X = 2$) fails. You can exploit this behavior of \+ to perform existence tests. Goal \+ \+(A) is true if there exist values of the currently unbound variables that make A true. The difference between \+ \+(A) and just A is that bindings done while testing \+ \+(A) are undone before proceeding.

18.10. An example

Section 16.3.1 illustrates backtracking by solving a planning problem where a farmer needs to cross a river with his goat, wolf and cabbage. This section redoes that example using Prolog.

less(X, Y) is true if X is alphabetically before Y.
Mode: less(in,in).

```
less(cabbage, farmer).   less(farmer, goat).
less(cabbage, goat).     less(farmer, wolf).
less(cabbage, wolf).     less(goat, wolf).
```

greater(X, Y) is true if X is alphabetically after Y.
Mode: greater(in,in).

```
greater(X,Y) :- \+(X = Y), \+(less(X,Y)).
```

add(XS, Y, ZS) is true if $ZS = XS \cup \{Y\}$.
Mode: add(in,in,out).

```
add([], X, [X]).
add([X|Xs], X, [X|Xs]).
add([X|Xs], Y, [Y,X|Xs]) :- less(Y,X).
add([X|Xs], Y, [X|R]) :- greater(Y,X), add(Xs, Y, R).
```

remove(XS, Y, ZS) is true if $ZS = XS - \{Y\}$.
Mode: remove(in,in,out).

```
remove([], _, []).
remove([X|Xs], X, Xs).
remove([X|Xs], Y, [X|Xs]) :- less(Y,X).
remove([X|Xs], Y, [X|R]) :- greater(Y,X), remove(Xs, Y, R).
```

Figure 18.7: **Operations on sets.** Sets are assumed to be lists in alphabetical order. Since we plan to use cuts and negation, the intended modes of predicates are important. Predicate less is defined directly, but in ISO Prolog you can also use operator @< to compare two symbols. For example, cabbage @< farmer succeeds.

One issue is representation of sets. We store a set as a list of its members. But it is important to keep the lists in a standard order, since otherwise it will be awkward to tell whether two sets are the same. The four things (cabbage, farmer, goat and wolf) are represented as symbols, and sets are stored in alphabetical order. Figure 18.7 shows operations on sets.

Next, as in Section 16.3.1, we need a way of representing a state, and a way of testing whether a state is admissible. A good choice for a state is a tree of the form state(L,R) where L is a set of the things on the left bank and R is the set of things on the right bank. Figure 18.8 shows operations on states.

Predicate 'follows' of Figure 18.9 chooses a state to go to next, and Figure 18.10 shows predicates that find a solution to the problem as a list of states.

18.11. Modifying a Prolog program on the fly

Prolog allows you to alter the axioms as the program runs. To add a positive literal C to the program, as a new axiom, perform goal asserta(C) (to add C as the first axiom)

admissibleBank(Bank) is true if nothing is being eaten in Bank.
Mode: admissibleBank(in).

```
admissibleBank(Bank) :-
  member(farmer, Bank), !.
admissibleBank(Bank) :-
  \+(member(goat, Bank)), !.
admissibleBank(Bank) :-
  \+(member(wolf,Bank)),
  \+(member(cabbage,Bank)).
```

admissible(S) is true if nothing is being eaten on either bank in state S.
Mode: admissible(in).

```
admissible(state(LeftBank,RightBank)) :-
  admissibleBank(LeftBank),
  admissibleBank(RightBank).
```

Figure 18.8: **Testing whether a state is admissible.** The cuts in the definition of admissibleBank are important. There are three ways that a bank can be admissible, and we do not want to try a given state three times just because of that.

follows(S, T) is true if state T can immediately follow state S in a solution.
Mode: follows(in,out).

```
follows(state(Left1, Right1), state(Left2, Right2)) :-
  member(farmer, Left1),
  member(Possession, Left1),
  remove(Left1, farmer, A),
  remove(A, Possession, Left2),
  add(Right1, farmer, B),
  add(B, Possession, Right2).
follows(state(Left1, Right1), state(Left2, Right2)) :-
  member(farmer, Right1),
  member(Possession, Right1),
  remove(Right1, farmer, A),
  remove(A, Possession, Right2),
  add(Left1, farmer, B),
  add(B, Possession, Left2).
```

Figure 18.9: **Finding a following state.** This is similar to the next function in Section 16.3.1, but it is a predicate.

lengthen(Sol, Longer) is true if Longer = [S | Sol] where S is an admissible state that does not belong to Sol and that can follow the head of Sol. That is, Longer is a slightly longer partial solution.
Mode: lengthen(in,out).

```
lengthen([St | Rest], [NewSt, St | Rest]) :-
  follows(St, NewSt),
  admissible(NewSt),
  \+(member(NewSt, [St | Rest])).
```

A list of states L is a partial solution if, (1) no state occurs twice in L, (2) each state in L is admissible, and (3) If $L = [\ldots, S, T, \ldots]$, then follows($T$, S) is true. (So the state sequence is backwards.) Suppose that Partial is a partial solution. Then solution(Partial, G, Sol) is true if Sol = A ++ Partial for some list A, Sol is also a partial solution, and the head of Sol is G.
Mode: solution(in, in, out)

```
solution([Goal | Rest], Goal, [Goal | Rest]).
solution(PartialSolution, Goal, FullSolution) :-
  lengthen(PartialSolution, LongerSolution),
  solution(LongerSolution, Goal, FullSolution).
```

```
showSolution([]).
showSolution([state(Left, Right) | Rest]) :-
  write(Left), write(' '), write(Right), nl,
  showSolution(Rest).
```

```
run :-
  EmptyBank  = [],
  FullBank   = [cabbage, farmer, goat, wolf],
  FirstState = state(FullBank, EmptyBank),
  GoalState  = state(EmptyBank, FullBank),
  solution([FirstState], GoalState, ASolution),
  reverse(ASolution, FwdSolution),
  showSolution(FwdSolution),
  nl.
```

Figure 18.10: **Solution of the Farmer-Wolf-Goat-Cabbage problem.** This is similar to the direct backtracking approach in Section 16.3.1.

```
dynamic seed/1.
seed(25).
random(N) :-
  seed(N),
  retract(seed(N)),
  R is (91*N + 3) mod 4096,
  asserta(seed(R)).
```

Figure 18.11: **A Prolog predicate random(N)**, which sets N to a pseudo-random number from 0 to 4095. It uses the seed predicate to simulate a global variable.

or assertz(C) (to add it as the last axiom). For example, asserta(animal(kangaroo)) adds axiom animal(kangaroo) to the program. You can remove C by performing goal retract(C). When C contains variables, retract(C) retracts all axioms that can be unified with C. The actions of asserts and retracts are not undone by backtracking; no information about them is written into trail frames.

One use of asserts is to add *lemmas* during computation. If a difficult goal is proved, it might be added to the axioms so that it will not need to be proved again. That use of asserts is quite benign and reasonable.

Asserts and retracts can also be used, however, to treat the program itself as a shared data structure through which a program can keep track of where it is, or to allow one part of a program to communicate with another part. As you can imagine, unfettered use of the capability to change a program while the program runs can lead to chaos. Therefore, ISO Prolog does not allow you to modify just any predicate on-the-fly. Before you try to add or remove an axiom for predicate p, you need to perform pseudo-goal `dynamic` p/n, where n is the number of parameters that p takes.

Figure 18.11 shows a simple pseudo-random number generator. It wants to update a global variable each time a new number is requested. But, since Prolog does not support global variables, it uses predicate seed to simulate one, by having just one axiom, seed(n), when the value of the seed is n. Goal seed(N) binds N to the current value of the seed by matching that axiom. Changing the seed value requires retracting the old axiom and asserting a new one.

A pseudo-random number generator is a reasonable application of a global variable. But you can imagine other uses. Figure 18.12 shows an example of a predicate called printnums that uses asserts and retracts to treat predicate count as a global variable. In an imperative program you are encouraged to avoid global variables. Using the program itself to simulate global variables can only be worse, and, as expected, the definition of printnums is very difficult to understand. Try working out a more principled definition of printnums.

18.12. Exercises

18.1. Write a Prolog definition of predicate prefix(X, Y), which is true if list X is a prefix of list Y. X is a prefix of Y if there exists a list Z such that $X ++ Y = Z$, where $++$ is concatenation. (A list is a prefix of itself.)

```
dynamic count/1.
printnums(N) :-
  asserta(count(1)),
  repeat,
  body,
  count(N),
  !.
body :-
  count(K),
  write(K), nl,
  next is K+1,
  retract(count(K)),
  asserta(count(next)).
repeat.
repeat :- repeat.
```

Figure 18.12: **A Prolog predicate printnums(N), which prints the numbers from 1 to N, one per line.** It uses asserts and retracts to treat the count predicate as a global variable that can be modified. Predicates body and repeat have no parameters. Repeat acts as a springboard; each time computation backtracks into it, repeat pushes it back forward again. Each iteration or printnums encounters goal count(N), which fails (back to repeat) if the count is not yet N. After successfully completing the loop, the cut removes the backtrack control frame that was created by repeat. This is intended to show what you *can* do, not to advocate this as a good way of programming in Prolog. (The repeat predicate is actually built into ISO Prolog, but its definition is shown for clarity.)

18.2. Write a Prolog definition of predicate suffix(X, Y), which is true if list X is a suffix of list Y.

18.3. Write a Prolog definition of predicate csublist(X, Y), which is true if list X is a contiguous sublist of list Y, that is, if there exist lists A and B such that $A ++ X ++ B = Y$.

18.4. Write a Prolog definition of predicate sublist(X, Y), which is true if list X is a (not necessarily contiguous) sublist of list Y, that is, if it is possible to obtain list X by removing zero or more members of list Y. For example, sublist([1,3,5], [1,2,3,4,5,6,7]) is true. Order must be preserved, so sublist([1,2], [3,2,1]) is false.

18.5. (Requires elementary calculus) Think about how to represent expressions in Prolog. Allow the expressions to use addition, subtraction, multiplication, the symbol x, and numbers. Write a predicate derivative(E, D) that is true if D is the derivative of E with respect to x. Your definition should be usable at least in the mode derivative(in, out). Does it work in other modes? Is derivative(2*x, 2) true? Be careful.

18.6. Write a definite clause grammar using ISO Prolog to determine whether a list is a sentence according to the grammar in Figure 17.15.

18.7. Modify your solution to the preceding problem so that your parser produces a parse tree of a sentence.

18.8. Definite clause grammars allow you to perform computations during the process of parsing. Typically, those computations handle issues that are difficult to deal with in a context-free grammar. Without changing the underlying context-free grammar, write an ISO Prolog definite clause grammar for the grammar of Figure 17.15, but ensure that any sentence that uses the word **meets** is not allowed to have identical subject and object. For example, [moe, meets, moe] should not be an acceptable sentence.

18.9. Write a Prolog definition of predicate pow(X, N, P), which is true when $X^N = P$, and which is intended to be used in mode pow(in, in, out). Assume that N is a positive integer.

18.10. Write a predicate sort(A, B) that is true when B is the result of sorting list A into ascending order. Assume that A and B are lists of numbers. The mode is sort(in, out).

18.11. Show the information flow in the axioms for factorial shown in Figure 18.4, in the mode factorial(in, out).

18.12. Show the information flow in the first axiom of follows in Figure 18.9.

18.13. Suppose that the first axiom for factorial in Figure 18.6 is replaced by

```
factorial(0,1) :- !.
```

without changing the second axiom.

(a) Does goal list (factorial(0, X), X = 2) fail, as it should?

(b) Does goal factorial(0, 2) correctly fail, as it should?

18.14. Prolog predicate var(X) succeeds when X is an unbound unknown and nonvar(X) succeeds when var(X) fails. Write a definition of predicate sum(X, Y, Z) that is true when $Z = X + Y$, and that can be used in any of the modes sum(in, in, out), sum(in, out, in) and sum(out, in, in).

18.15. Write axioms for predicate cousin(X, Y), which is true if X and Y are cousins. That is, X and Y must have a grandparent in common, but they must not also have a parent in common or be the same person. Use predicates parent(X, Y) (X is Y's parent) and grandparent(X, Y) (X is Y's grandparent). Design this predicate to run in mode cousin(in, in).

18.16. Does your definition of cousin(X, Y) from the preceding exercise work in mode cousin(in, out)? If not, can you make it work in that mode? Either give a definition that works in that mode, or argue that doing so would be very difficult.

18.17. Write a Prolog program to solve the missionary/cannibal puzzle (Exercise 16.12). Use jailbird notation to represent numbers.

18.18. How are the concepts of negation as failure and the closed world hypothesis related? Is one based on the other, or are they completely unrelated ideas?

18.19. Explain why using negation as failure typically limits the mode of a predicate. Does negation as failure always work when all of the parameters are in-parameters, containing no variables? Explain why or why not.

18.20. Show how to translate Prolog axiom `P(X) :- A(X), (Q(X) ; R(X))` into a form that does not use a semicolon in two ways, one using an auxiliary predicate that stands for the disjunction (Q(X) ; R(X)), and another that does not introduce any auxiliary predicate. Does it make a difference which one is used? What if there is a cut, as in `P(X) :- A(X), (Q(X), ! ; R(X))`? Try this in a version of Prolog that is available to you and try to determine which one is used in that version.

18.13. Bibliographic notes

Clocksin and Mellish [28] describe ISO Prolog. Colmerauer and Roussel [29] give a history of Prolog. Marriott and Stuckey [68] describe languages that extend Prolog by allowing more kinds of relationships among values than equality (unification).

Warren [99] describes an abstract machine suitable for use in Prolog implementations.

Clark [27] discusses negation as failure, and Naish [73] shows how to make it mathematically correct by deferring negations until they can be done. Constraint programming languages [38, 68] do much more, allowing a variety of conditions to be asserted and held until it is possible to test them.

Chapter 19

Object-Oriented Programming

19.1. Object-based programming

19.1.1. The object-based view of abstract data types

An abstract data type is characterized (partly) by the operations that form its signature, so those operations are thought of as part of the type. For example, if a Queue type supports operations Insert, Delete, front and isEmpty, then you would say that the type has those four operations.

But there is an equally valid *object-based* point of view, where the operations are associated not with the data type but with the data items, or *instances*, themselves. In the object-based view, each individual queue instance, called an *object*, supports the queue operations, and is thought of as carrying its operations with it. Each object decide how it performs operations. It is possible, for example, to have one object that implements Insert, Delete, front and isEmpty as a queue would, with insertion at the back, and another that implements them the way a stack would, where new insertions are at the front. From the outside, they look the same, because they support the same operations, and the two objects can be used in similar ways. That provides a programmer with a high degree of flexibility and makes it possible to make a subprogram have a variety of different behaviors by varying the characteristics of the objects that the subprogram uses.

Because object-based programming associates operations with objects, not with types, it is able to achieve some of the same goals as abstract data types, but to dispense with the notion of a type, if desired, allowing the programmer to concentrate on the objects themselves. That is not to say that object-based programming is incompatible with types; rather, types are optional.

In object-based terminology, the functions and procedures that an object holds are called the *methods* of the object.[1] But an object is more than a collection of methods; it also needs to remember information. For example, a queue object needs to know what is in the queue at the moment, which it remembers in a collection of variables. The current set of values of the variables in an object, taken as a whole, is called the *state* of the object, telling exactly what information an object is storing at a given point in time. Figure 19.1

[1] The term *method* is probably not the best choice, since the intent of a function is to hide, not to expose, the method that is employed by and encapsulated within the function. But the terminology has become standard, so we use it.

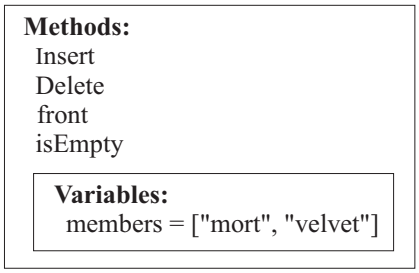


Figure 19.1: **A snapshot of a queue object.** The methods (Insert, Delete, front and isEmpty) are visible to other objects, but variable 'members' is private, so is only visible to the methods inside this object.

shows a view of a queue object holding values "mort" and "velvet" in its state, and also holding the queue methods.

Various characteristics of a type can be brought down to the level of objects. For example, an abstract data type has a carefully chosen set of operations, its *signature*, that it supports. By the same token, an object interacts with its surroundings in carefully controlled ways, through operations that are known as the *public* methods of the object. The state of an object is hidden from view, visible only to the object itself, in just the way that the representation of a fully abstract data type is hidden from view behind the name of the type. That is illustrated in Figure 19.1 by showing the state encapsulated inside the object. The state is *private* to the object, and can only be seen from outside the object indirectly through the public methods.

19.1.2. Agents, components and program organization

Focusing on objects suggests a different way of thinking about programs. In a procedural program, you think of performing a procedure on a variable or variables. But in object-based programming, you think of asking an object to perform an operation. The object is thought of as an *agent* capable of performing requested actions.

An object resembles a miniature computer. It has memory and some software that uses that memory. An object-based program can contain thousands of tiny computers communicating with one another, each one specialized at performing a particular task. An object-based programmer thinks, in some ways, like a computer network designer, placing different services in different machines, storing information on the computer that needs it, and arranging for machines to communicate with one another.

The difference in focus, from a procedure operating on an object to an object performing a procedure, is a little like walking around a statue and looking at it from another side. It might sound slight, but it can lead to a profound difference in the way programs are written. Procedural software design typically begins with a decision on which procedures need to be implemented, with the data flow among procedures following later. Since you start by concentrating on the procedures, you tend to group similar procedures together, and that leads to programs that are organized around procedures and algorithms. For example, all procedures that print something might be put together into a single module, while procedures for mathematical operations would be put into a different module.

In object-based design, the designer starts by thinking about the objects (the miniature computers) that the program will support, and what information they will remember and manipulate, with the methods following later. The focus is on the information that each object will be responsible for maintaining. When the methods are eventually written, it is natural to put all of the methods that belong to a particular object or group of related objects in one module. That leads to programs that are organized around data and types of data. For example, all operations that affect queues are put in one place, regardless of what those operations do. An operation for printing the content of a queue would naturally go with other operations that belong to a queue object, not with other operations that print things.

19.1.3. Building objects

The most obvious way to build an object is in a custom way, choosing each variable and method to put into the object, giving each a name, and determining which are public and which are private. But not all objects need to be built that way.

One way to create a new object is to make a copy of an object that you already have. If an object did not contain variables that could be changed, there would be little point in making a copy. But because an object can change its variables, two different copies of an object can go in different directions, modifying their states in different ways.

Extension and inheritance

If you have an automobile, sometimes you want to add options. For example, if you move from a cool climate to a warmer climate, you might discover that you need an air conditioner. You take the car to a dealer and have one installed. Objects in a program can similarly be extended by adding new variables and methods. The larger object is said to *inherit* features from the object that it extends.

Overriding methods

Sometimes, when you extend an object, you want to replace some methods rather than inheriting them. For example, suppose that a new queue object is to be created from an old one, but the new queue operation is intended not only to perform operations, but to keep track of the total number of Insert operations that have been done on it since its creation. The new object will need a new variable to hold the insertion count, and it will need to replace the Insert operation by one that not only does the insertion but also increments the counter. When a method is replaced, the replacement is said to *override* the original method.

Sometimes an overriding method wants to make use of the method that it is overriding. For example, a method that both inserts and adds one to a counter probably wants to make use of the Insert method that would have been inherited. The new Insert function can be built by getting (not running) the Insert method from an object and then building a new function around that method.

19.1.4. Polymorphism in object-based programming

A function that uses an object will ask that object to perform certain operations, and it only requires that the object supports all of the operations that are actually used. Suppose,

for example, that the object is asked to perform operations "Insert", "front" and "Delete", and nothing more. Then the function will work on any object that supports *at least* those operations. It will work on a queue object, or on an augmented queue object (with additional variables and methods), or even possibly an object that works very differently from a queue object, such as a stack or set object, as long as it has the required methods. Functions and procedures that use objects are, therefore, automatically polymorphic.

19.1.5. Transition to object-oriented programming

Unfortunately, direct implementation of object-based programming has some drawbacks. One concerns efficiency. If objects carry their methods with them, then an object that holds many methods must occupy quite a bit of memory. When an object is copied, that memory is copied as well.

Another problem concerns just how methods will be written and attached to the objects. Although functional languages support functions as data items, most other kinds of languages typically do not do that in any but the most limited of ways. But if a function is not a value, what is being stored inside the object as a method?

Object-oriented programming introduces a notion of a *class* that is similar to a type. Instead of associating functions with objects, we move them back to the level of classes. That appears to backtrack and to lose the advantages that object-based programming affords. But that is not really the case. It is possible to *imagine* the operations as being kept with the objects when, in fact, they are kept with the classes. Object-oriented programming offers the convenient fiction that it is object-based programming while keeping the advantages that the traditional view of abstract data types offers.

19.2. Classes and object-oriented programming

A class can be thought of as a template or plan for constructing an object, along with some ways of building objects that follow the plan. Each object constructed by class C is called an *instance* of class C, and has a tag attached to it indicating that it belongs to class C. A class definition consists of a collection of variable and method definitions describing characteristics of the instances. For example, class Bucket might indicate that each of its instances has a variable called *volume* of type *Liter*, and a method called *Grow*. The class definition gives the details of how Grow works. Since the variables and methods described by a class are actually characteristics of an instance of the class (using the object-based viewpoint), they are called *instance variables* and *instance methods*. All instances of a class share the same method implementations, but each object has its own copies of the instance variables.

Figure 19.2 illustrates, defining a class that describes the instance variables and instance methods of a queue. The syntax is not that of any particular language, although we use some syntax from Cinnameg for constants and expressions. For simplicity, the members of a queue are strings. A queue object holds a list of strings as its state and has methods for inserting a string (at the back), deleting a string (from the front), getting the front string, and checking whether the queue is empty.

A class actually has several different aspects to it. It plays the role of a concrete type by indicating how information is represented and how methods are implemented. It plays the role of an abstract type by indicating a signature, and is part of a form of polymorphism that

```
class Queue
  instance variables:
    members: [String]
  instance methods:
    Insert(s)
      members := members ++ [s]
    Delete()
      members := tail(members)
    front()   = head(members)
    isEmpty() = members == []
end class
```

Figure 19.2: **A queue class.** The syntax is generic, not that of any particular language.

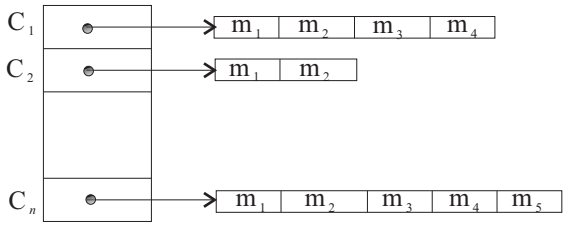


Figure 19.3: **An implementation of a dispatch table, with a row for each class and a column for each method.**

is discussed in the next chapter. Classes are responsible for building the objects that they describe, so they play the role of object factories. A class also takes on the responsibilities of a module, controlling visibility of components and scopes of names. We look at aspects of classes separately.

19.2.1. Getting components of an object: the dispatcher

The state of an object, represented by the values of its variables, is stored with the object. The instance methods, however, are stored elsewhere. When object A needs to run method m, the runtime support first gets the class of A from A's tag. Typically, each class is given a number, and each method within a class also has a number. A table, called the *dispatch table*, stores an array of pointers to methods for each class, as shown in Figure 19.3. Suppose class Queue is assigned number 4 and the Delete method is the second method in the class. Then a queue object finds its Delete method by looking in row 4 of the dispatch table, and getting in the second entry in the array that is found there. Two array fetches (one for the class and one for the method) suffice to get a method from the dispatch table, so looking up a method is cheap.

```
class Queue
  instance variables:
    members: [String]
  constructors:
    newQueue()
      members := []
    newQueue(init: [String])
      members := init
  instance methods:
    as in Figure 19.2
end class
```

Figure 19.4: **The Queue class with two constructors.** The name newQueue is overloaded.

19.2.2. Constructing objects

In addition to describing the components that objects have, a class can take on the role of an *object factory*. When an object is created it is important to put values into its instance variables, since otherwise the object is in an undefined state. The initial state of an object is set up by a *constructor*. Ways of expressing constructors differ from one language to another. Figure 19.4 shows a generic approach to constructors.

When you order a car from a factory, you should be able to request options such as paint color and air conditioning. Objects can also be created with different options, and it is common for a class to have more than one constructor, allowing you to order objects with various initial configurations. To illustrate, Figure 19.4 shows the queue class with two constructors, so that

```
q := newQueue();
```

makes q a new instance of class Queue that is initially empty, and

```
q1 := newQueue(["cat", "dog"]);
```

makes $q1$ a new queue initially holding two strings.

19.2.3. Classes as modules

Modules are used not only to group things but also as a way of hiding some of them so that, while making modifications to a module, you know that none of its private things are visible in any other module. In object-based programming objects also possess notions of visiblity, so they share some characteristics of modules.

Object-oriented programming is heavily influenced by ideas of object-based programming. But, having moved method definitions from objects to the larger level of the class, it makes sense to move issues of privacy to that level too. That has an important consequence; each object can see private things inside other objects of the same class. For example, suppose that you want to augment the Queue class by an operation that concatenates two queues together. The concatenate operation is performed by one queue, the agent, with another queue q as a parameter. In object-oriented programming, the implementation of

the concatenate method can use not only the private variables of the agent queue, but also those of the parameter queue q. That would not be possible in object-based programming.

Is the ability of one object to see private aspects of another reasonable? A good approach to answering that question is to ask what is gained and what is lost. Certainly, something is gained. Efficient and sensible concatenation of queues will almost surely require direct use of the representations of both queues, and the object-oriented view of privacy at the class level makes that possible. But losses are small. Remember that the main point of making things private is to ensure that you do not need to look outside of a module when making modifications to private things. The privacy constraints decouple different modules, making a program more flexible for modification. But since the class is the module, and all objects of a given class share the same implementation, allowing one object to make use of private things in another object of the same class does not make modification much more difficult. You still need to examine the entire class, and nothing more, when you make modifications to its private variables and methods.

Indicating visibility

A simple way to indicate visibility of components, employed by Java, is to mark each thing in a class as private or public. Another way to indicate visiblity is to separate a class into two parts, one describing the public aspects and the other telling additional (private) information. Only the part of the class that describes public features of the class is put in a place where it can be read by other modules. Cinnameg uses that approach.

Programming languages strive for ideals, but are often limited by practical considerations. In order to allocate memory for an object of a given class, it is often necessary to know how much space is needed, and that requires knowledge of information about the variables in the class. When positional inheritance is used, each subclass needs to know how much space is used by its base class in order to know where its own variables fit into the representation, and how to implement the selectors. Assigning numbers to methods also requires knowledge of the number of methods in the base class, including private methods. A consequence of those requirements is that, while compiling one class, a compiler typically needs to know private information about other classes.

Languages like Java that only employ visibility annotations make all information available to a compiler. Some languages, such as C++, allow you to separate a class into interface and implementation parts, but require you to list some information about private variables and methods in the class interface (marking them private) so that a compiler can see that information. Cinnameg avoids the necessity of adding private information to the class interface by calling functions, implicitly defined with the implementation part of a class, that provide the necessary information at run time. That is a more expensive way of doing things than what is done by C++ or Java, since information about sizes needs ends up hidden inside subprograms that perform object construction; when a compiler knows all of the information, it can expand out construction details, avoiding calls to subprograms. There are often tradeoffs between ideals and efficiency, and language designers need to decide where their priorities lie.

19.2.4. Class components and classes as objects

Occasionally, a programmer needs a variable that is shared by all instances of a given class. For example, you might want to keep track of the total number of members of all queues

of a given kind, using a variable that is shared by all queues, and that the Insert and Delete operations can increment and decrement. Some classes also need to have constants. For example, a class that is responsible for conversions between different distance measures probably needs to know that an inch is 2.54 centimeters. It does not make sense to put a copy of a constant in every object; the value is the same, and storing it in every object is wasteful of memory. Instead, a single copy of each constant should be shared by all of the objects.

Variables that are shared by all instances of a given class are called *class variables*, to distinguish them from variables that belong to the instances of the class. Similarly, a class constant is stored with the class, and is accessible to all instances of that class. Methods that belong to a class, rather than to instances of the class, are called *class methods*.

An obvious way to handle shared components is to associate class variables, constants and methods with the module of a class. Java takes that approach. (In Java, you label class variables and methods **static**.)

Classes as objects

There is another idea for handling class variables, constants and methods that is both versatile and intuitively appealing, illustrated by Smalltalk. Think of a class as an object factory. The factory itself is also a kind of object, and can have variables and methods. When you create class variables and methods, you are simply populating the class object with components. When you use a class method, you ask the class object to perform the method, just as you would ask any other object to perform a method.

One of a factory's methods produces a new object. So the constructors of a class are really part of the class object, not part of the instances of a class. (An automobile does not possess the capability to build a new automobile.) To create a new instance of a class, you ask the class object to run one of its constructor methods.

An advantage of making classes objects is that they become first-class citizens; you can pass a class as a parameter to a method, return a class from a method, or store a class object in a variable. Of course, if a class is an object, you expect it to be an instance of some other class, such as a class of all object factories. Clearly, treating a class as an object has implications for the language design; Smalltalk's solution to these issues is discussed in Chapter 21.

19.3. Object-oriented languages

Object-oriented programming languages provide some means of describing classes and creating objects that are instances of those classes. They also must provide a means of asking an object to perform a method. Java uses syntax *agent.action*. Cinnameg uses *agent*'s *action*; the action starts with a method, and you think of the method as belonging to the object. Smalltalk dispenses with extra syntax; you just write *agent action*. But those differences are minor compared to more substantive language differences.

Some object-oriented languages, such as Java, treat a class as a type, and their compilers perform some (static) type checking. But it is also possible to provide classes in a typeless language. Just because a program is based on classes does not mean that a compiler needs to remember what type of thing is in a given variable, or to perform any type checking. Remember that each object already carries with it a tag indicating its class, so it is a fairly simple matter to perform run-time type checking. Smalltalk is an

example of a typeless object-oriented language. Classes in Smalltalk are used for describing characteristics of objects and for building objects, but they are not used for any kind of compile-time type checking. Although Smalltalk is object oriented, it has the feel of an object-based language, where you can use any object in any context, as long as the object has all of the methods that are used in that context.

19.4. Exercises

19.1. How is an object like an abstract data type?

19.2. What is inheritance in object-based programming?

19.3. What does it mean to override a method?

19.4. Explain why a function that takes an object as a parameter is polymorphic. What kinds of objects can it use?

19.5. For this problem, ignore the issue that some components are public and some are private. Suppose you decide to use the idea of records for types of objects. An object with components a: Integer and b: String $\rightarrow$ Integer has record type $\{a$: Integer, b: String $\rightarrow$ Integer$\}$. You would like to keep the polymorphism that object-based programming enjoys. Define a relation $\leq$ on record types so that, if f takes a parameter of type B and $A \leq B$, then you should be allowed to apply f to a parameter of type A. (If you think of an object of type B as representing a mammal and an object of type A as representing an anteater, then you should be able to pass an anteater to a function that wants to receive a mammal.)

19.6. What is an advantage of storing instance methods in the dispatch table rather than with the objects?

19.7. Why is a notion such as a class necessary for using the dispatch table?

19.8. Since instance methods are not stored inside objects in object-oriented programming, how do you think an instance method gets access to the object (self) to which it supposedly belongs? Describe a way in which an implementation might do that.

19.9. What is the purpose of a constructor?

19.10. Would it work to store the dispatch table as an array of linked lists? What would be the advantage or disadvantage of doing that?

19.11. Why is it important for an implementation to use integer tags instead of using the class name (a string) as a tag? What would be the consequences of tagging items by a string? Keep in mind that, every time any instance method wants to use another instance method, it needs to do a lookup in the dispatch table.

19.12. Using any reasonable syntax, write an implementation of a class whose instances represent DVDs at a rental store. Each DVD has to remember its state (rented or on the shelf) and, if rented, it has to remember who rented it, and when it is due back. How can you represent the information? What instance methods should each object have?

19.13. Parameters to methods can be simulated, to some extent, by instance variables. For example, if you have several methods that take a string as a parameter, then you can remove that parameter, add an instance variable (theString), and add a method to set theString to a given string. To use a method, you first set theString to your desired string, and then call the method, which just looks in theString to find out the value of the desired parameter. Would you recommend that approach? Why or why not?

19.14. What is the difference between a private variable and a protected variable?

19.15. Does it make sense to have a protected method? Give an example.

19.16. Object-based and object-oriented programming have different notions of visibility. What is an important difference between the two, and what causes the difference?

19.17. Do you have to create an instance of class C in order to use one of C's class methods? If so, why? If not, which object would you ask to perform the class method?

19.18. Which objects should be able to use a private class variable of class C? Try to think of all kinds.

19.19. Can a class method refer to instance variables or methods of the same class? Why or why not?

19.20. Does it make sense to have a protected class variable? Explain. Where would it be visible?

19.21. Suppose that class B is a subclass of A, and that class A defines a protected instance variable V. An instance method m defined in class B takes a parameter, y, of (nominal) class A. Should m be allowed to use y's variable V? Consider the pros and cons, assuming that you are making the choice for a new programming language. Keep in mind that A might have other subclasses.

19.22. This continues the ideas of the previous question. Suppose that B and C are two subclasses of class A, and that class A defines a protected instance variable V. By inheritance, objects of class B and C all have variable V. Should a method that is part of an object of class B be able to use the variable V that belongs to another object of class C? Support your conclusion.

19.23. Show that, if checking of what is allowed is to be done at compile time, and if your answer to question 19.21 is that an object of subclass B should be able to see protected variables of objects of class A, then for all practical purposes, your answer to question 19.22 is yes, because an object of class B can already see the variable V that belongs to objects of class C. How can a class B object do that? Remember that you are allowed to store an object of class B or C into a variable of type A.

19.5. Bibliographic notes

Abadi and Cardelli [3] describe a theory of object-based programming, and Wegner [100] discusses object-based languages. Kay [58] discusses the idea of an object-based program as a collection of small, interacting computers in the context of the development of Smalltalk.

Castagna [23] explains object-oriented programming. Cook [31] compares object-oriented programming with abstract data types.

Chapter 20

Inheritance and Polymorphism

20.1. Polymorphism

A class C describes a *signature* consisting of the variables and methods that all of its instances possess, and an instance of class C possesses exactly those characteristics, no more and no less. But imagine *using* object, by employing the public features that it possesses. From the point of view of a part of a program that interacts with a queue object, the object must have *no less* than the features mentioned in the signature. It does not really matter if it has additional features. That suggests that methods that use queues, for example, can be polymorphic, working on any type of object that possesses at least the signature of queues.

20.2. Inheritance

But there is more to polymorphism. In Section 19.1 we saw that you can create a new object by extension, inheriting methods from an existing object. In object-oriented programming, you do a similar thing by letting a class inherit methods from another class. The methods of class C are inheritable because they are designed to be polymorphic over all classes that inherit methods from class C. That allows you to reuse methods, avoiding the need to write them again for each new class. Figure 20.1 shows a definition of class LQueue using inheritance, where an object of class LQueue has not only all of the Queue operations but also has one that yields the current length of the queue.

To implement inheritance, the row of the dispatch table for Queue is copied into the row for LQueue, so that instances of class LQueue end up using the Queue methods to perform Insert, Delete, isEmpty and front.

The *class hierarchy* indicates how classes relate to one another through inheritance. Class A is a *base class* of B if B inherits directly from A. For example, the base class of LQueue is Queue. Class B is a *subclass* of class A if it inherits from A either directly or indirectly. For example, if A is the base class of B and B is the base class of C, then C is a subclass of A. Class A is a *superclass* of B if B is a subclass of A.

```
class LQueue extends Queue
  instance methods:
    length() = length(members)
  constructors:
    newLQueue()
      super()
    newLQueue(init: [String])
      super(init)
end class
```

Figure 20.1: **Class LQueue, a subclass of Queue.** The syntax is generic. For the time being we are not concerned with issues of privacy, and allow class LQueue to use what would normally be a private component, variable *members*, of class Queue.

Each constructor needs to run a constructor in the base class (Queue) to initialize the inherited variables, indicated in our generic syntax by a pseudo-constructor called super.

20.2.1. Constructors in subclasses

Class LQueue needs a constructor that initializes the inherited variable, *members*. But class Queue already has such a constructor, and it is sensible to use Queue's constructor to do the initialization. In fact, in cases where some of the superclass's variables are private, using a constructor in the superclass is the only available way to initialize them. The problem is that we want to run the Queue constructor on an object of class LQueue, even though the Queue constructor normally creates an object of class Queue. Object-oriented languages have various ways of dealing with that. In our generic syntax, we write super(p) to call the constructor in the base class with parameter p. Figure 20.1 shows class LQueue with two constructors.

20.2.2. Inheritance by position

Inheritance is enabled by the polymorphism of a class's method definitions. For example, class LQueue gets its front method implicitly because that method is polymorphic over all subclasses of Queue. But how is that polymorphism achieved?

An object is typically represented as a block of memory holding the object's tag and its instance variables. Suppose that we arrange for each variable to be put in a particular place in the representation. For example, the tag might come first, followed by the first variable, then the second variable, etc. Each method gets a variable by looking in its position in the block of memory that represents the object.

A class inherits variables from its base class, which means that an instance of class LQueue has a variable called members. If we simply arrange for it to be in the same position as it would be in an instance of class Queue, then the methods of class Queue will work on the representation of an instance of class LQueue because they find the members variable in the expected position. We call this idea *inheritance by position*.

Suppose that class LVQueue, also a subclass of Queue, introduces a new variable called len that holds the length of the queue. In order to use inheritance by position, we need to keep variable members in the same location as it is in class Queue. But that is easy to achieve simply by adding variable len to the end of the block of memory, as shown in

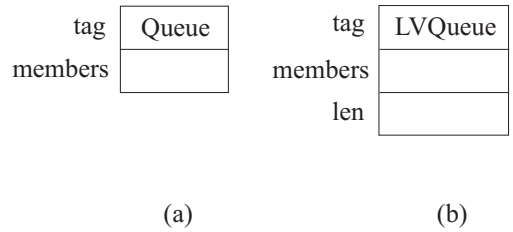


Figure 20.2: **The representations of two objects, (a) an object of class Queue and (b) an object of class LVQueue, with an additional variable.** The members variable is the second variable, counting the tag, in both cases.

Figure 20.2.

A program employs a *selector* to find a variable, with a positional selector simply looking at a given spot in a block of memory. There is another kind of selector at work in the implementation of an object-oriented language that is responsible for finding methods by looking in the dispatch table. In order for a method definition to be polymorphic, the method selectors that it uses (to find other methods) must be polymorphic. That is, they must work correctly in subclasses.

The dispatch table assigns a number to each method in a class. The idea that we have used for variables, to put each variable in a fixed position within the memory that represents an object, can also be used for methods, by putting each method in a fixed position in a row of the dispatch table. The Insert method, for example, should be given the same number in class LQueue as it is given in class Queue. To achieve inheritance, it suffices to copy the entire row of class Queue into the row for class LQueue, and to put new methods of class LQueue at the end of the row.

20.2.3. Single and multiple inheritance

So far we have only thought about inheritance where a class has just one base class. A language that requires every class to have at most one base class is called a *single-inheritance* language. But there are cases where *multiple inheritance* is called for, in which a class can have more than one base class. Multiple inheritance is common in graphical systems. An object often needs to know information about what it represents, but also needs to be able to draw an image of itself and to interact with the user. For example, a class whose instances represent bears might need to be a subclass of Mammal and also a subclass of Graphics, so that a bear object can be used in contexts where it is asked to draw itself.

Multiple inheritance presents some serious problems to object-oriented languages. One problem is how variable and method selectors are managed. In a single-inheritance language, variables and methods in a subclass are added to the end of the variables and methods of its base class and selectors are automatically polymorphic. With multiple inheritance, that is not possible. The variables for the base classes will need to be combined, and it will not be possible to keep them in the same positions.

Another problem concerns determining what gets inherited. Suppose that classes Mammal and Graphics each define a method called grow. If an object of class Bear is asked to perform its grow method, which one should it use? Some sort of priority indications will

```
class LVQueue extends Queue
  instance variables:
    len: Integer
  constructors:
    newLVQueue()
      super()
      len := 0
    newLVQueue(init: [String])
      super(init)
      len := length(init)
  instance methods:
    length() = len
    Insert(s)
      members := members ++ [s]
      len := len + 1
    Delete()
      members := tail(members)
      len := len - 1.
end class
```

Figure 20.3: **Definition of class LVQueue with a new variable to hold the length of the queue.** The Insert and Delete methods are overridden, but the front and isEmpty methods are inherited.

need to be given in the program. So multiple inheritance has implications for the language as well as for the implementation.

Some object-oriented languages, such as Eiffel and C++, support multiple inheritance in order to allow the programmer freedom to relate classes in whatever way corresponds most closely with reality and with needs. But those languages are necessarily more complicated than languages that restrict the programmer to single inheritance, such as Smalltalk and Java, and we will stick to single inheritance here.

20.3. Overriding methods

After adding the len variable to class LVQueue, you need to ensure that len is incremented each time a value is added to the queue and it is decremented each time a value is removed from the queue. Unfortunately, the Insert and Delete methods inherited from class Queue do not do that, and must be replaced in the subclass. That is, their definitions must be *overridden* rather than inherited. Figure 20.3 shows class LVQueue with Insert and Delete overridden. But there is an unpleasant aspect of that definition. Notice that the bodies of Insert and Delete from class Queue have been copied and modified. But why should they be copied? It would be better to use the definitions from class Queue, but to perform some extra steps as well. That is more important than just a matter of additional work in writing the methods. If later you modify the method definitions in the Queue class, you do not want to be required to modify the definition of the LVQueue class as well.

```
class LVQueue extends Queue
  ...
    Insert(s)
      prior Insert(s)
      len := len + 1
    Delete()
      prior Delete()
      len := len - 1
end class
```

Figure 20.4: **An implementation of class LVQueue using prior methods.** Only the definitions of Insert and Delete are shown.

So an object of class LVQueue needs to be able to get the Insert method out of the dispatch table in the row for class Queue. Object-oriented languages generally provide a way to do that. In our generic syntax, we will write *prior Insert* to indicate the version of Insert that would have been inherited. A better definition of class LVQueue is shown in Figure 20.4.

Overriding methods is a versatile tool, but it must be used with caution. Generally, a method should keep its intent even when it is overridden. Otherwise it will not work as expected where it is used. Remember that a polymorphic method might think that it is working with a Queue object when what it really has is an LVQueue object. Fortunately, the overriding definition of Insert for class LVQueue still inserts a value into the queue.

20.3.1. Inheritance and libraries

Imagine that you are writing a graphically based program, and, as part of this program, you create a window that holds some text, editable by the user, that allows the user to type, backspace, select text with the mouse, copy text to the clipboard, change the font of text, etc. Typically, this kind of object needs to respond to a large number of events such as a key being pressed or released, the mouse moving over the window, and so on.

Obviously, that is just the sort of thing that you would like to have provided for you in a library, where all you have to do is create a standard kind of object and let the library deal with all of the possible events. But whoever writes the library cannot think of everything that you might want to have. For example, suppose that you want a control-I keystroke to change to an italic font, but the library class does not work that way. All you need to do is to create a subclass of the library class where your subclass handles the event of a key being pressed. If it sees control-I, it changes the font, and otherwise it runs the key-press event method that it would have inherited.

Libraries made up of classes tend to be versatile and, due to inheritance, modifiable, since programmers can create subclasses that inherit most of their behaviors from the library, and only explicitly say what they want to do differently.

20.3.2. Protected visibility

Using classes instead of objects as modules increases visibility from one object to another in some cases, but it also has another effect: it can decrease visibility within a single

object. We have classified each method or variable of an object as either public (accessible to everybody) or private (accessible only within the module of the object's class). Now bring the class hierarchy into the picture. Class LQueue of Figure 20.1 is derived from class Queue. If the *members* variable of Queue is private, then it is inaccessible in class LQueue, making it impossible to write the length method. Private variables are inherited, and every instance of class LQueue possesses them, but they can only be used in the class where they are created. So an object can contain variables (members) to which only some of its methods (those that are written in the module of class Queue) can refer.

From an object-based standpoint, where the objects themselves are thought of as the modules, it does not seem to make much sense to give some of the methods within an object special privileges not shared by other methods in the same object. But from the perspective of traditional modular software design, it makes very good sense to say that a particular concept (the existence of a variable called members, for example) can only be used in a single module, since then any change to that concept can only affect that one module.

So there are conflicting ideas about visibility. It is not clear whether privacy should be defined in an object-based or an object-oriented way. That is, should private variables and methods be accessible only in the module of a class, or in the module of that class and all of its subclasses? The solution that is typically adopted is to let the programmer decide, on a case-by-case basis, which notion of privacy to use. A variable or method that is marked *private* is accessible only in the module of the class in which it is defined. A variable or method that is marked *protected* is accessible not only in the class where it is defined, but also in the modules of all subclasses of that class.

20.4. Pragmatics of inheritance

Imagine that, in order to keep track of how much money a particular individual has on hand, you have created a class called Wallet. A wallet can receive money and pay out money, and remembers a variable, Cash, telling how much cash it contains.

Now you would like to create another class, Person, representing characteristics of a person who is modeled within your program. One characteristic is how much money this person has on hand. If you think in terms of the variables, you realize that this person needs to have a Cash variable. But class Wallet already has that variable, plus methods for managing it. If you make Person a subclass of Wallet, then every instance of class Person will automatically inherit the Cash variable. A Person instance also inherits methods to accept money and pay out money, which is reasonable, since those methods will be handled by the Wallet class. Handing some money to a person becomes equivalent to handing that same money to his or her wallet. This certainly sounds like an appealing way to do things.

Unfortunately, making Person a subclass of Wallet suggests that a person is a special kind of wallet, which is absurd. It allows you to put an instance of class Person into a variable of class Wallet. But doing so is, very likely, a mistake in the program, not something that a programmer would deliberately do.

The problem is that, instead of saying that a person *is* a wallet, you want to say that a person *has* a wallet. Well-principled use of inheritance suggests that you should only use inheritance for the case where something of class B *is* a special kind of thing of class A. Instead of making Person a subclass of Wallet, you add an instance variable of class Wallet to class Person. Of course, that means that methods that might have been inherited from Wallet need to be written for Person, but at least you have a more principled and sensible

design.

20.5. Type checking and polymorphism

We can express the notion of subclass as *partial order* among classes; we write $A \leq B$ if $A = B$ or B is a subclass of A.

In a typical statically typed monomorphic language, such as Pascal, the type checker makes sure that certain types match. For example, if you write expression $f(x)$, and f takes a parameter of type String, then the compiler ensures that x has type String; otherwise there is an error.

But object-oriented languages require more subtle type checking that allows the polymorphism of the class hierarchy to come through. Suppose that method f takes a parameter of class A. and suppose that expression $f(x)$ occurs in a program, where x has type B (another class). Then the compiler only has to ensure that $B \leq A$ rather than $B = A$. For example, suppose that function numberOfTeeth has polymorphic type Animal $\rightarrow$ Integer, where Animal is a class whose subclasses represent particular kinds of animals. If b has type BlackBear, then numberOfTeeth(b) should be well-typed, since b is a special kind of Animal. Similarly, if v is a variable that is supposed to hold an Animal, then a program should be allowed to store a value of type BlackBear into v.

20.5.1. Relationships among structured types

The concept of the class hierarchy can be extended to structured types. For example, if function g has type (Animal, Animal) $\rightarrow$ Integer, then g should be able to take an actual parameter of type (Giraffe, Lion). In general, say that $(A, B) \leq (C, D)$ when $A \leq C$ and $B \leq D$. With that definition of $\leq$ for tuple types, the compiler still just checks expression $f(x)$ by ensuring that the formal parameter type A and the actual parameter type B satisfy $B \leq A$.

The $\leq$ relationship can also be extended to lists and functions. For lists, it should be clear that $[B] \leq [A]$ just when $B \leq A$. For example, a list of bears is a special case of a list of animals, and you should be able to use a list of bears in a place where a list of animals is required. Since list types $[A]$ and $[B]$ are ordered in the same way as their member types, A and B, list types are said to be *covariant*.

Relationships among functions

Some languages allow functions to be treated as values, and passed as parameters to other functions, so it is important to know how functions fit into polymorphism. Functions exhibit a different kind of relationship from lists or ordered pairs. Imagine that class Number has subclass Integer. Suppose that you have two functions gen and spec, where gen has polymorphic type Number $\rightarrow$ Boolean, and spec has type Integer $\rightarrow$ Boolean. Since gen, the more general function, can take any kind of number as a parameter, it can be used any place where spec could (at least from the perspective of types). By the same reasoning that was used for pairs and lists, you should expect that (Number $\rightarrow$ Boolean) $\leq$ (Integer $\rightarrow$ Boolean). Recall that, in type $A \rightarrow B$, A is called the domain type and B is called the codomain type. Functions are ordered in the opposite order as their domains, and, in general, $(A \rightarrow B) \leq (C \rightarrow D)$ just when $C \leq A$ and $B \leq D$. Functions are covariant in

their codomains, but *contravariant* in their domains, since they have the opposite order as their domains.

Relationships among box types

Boxes (variables, Chapter 6) are a simple kind of object, and they need some thought. Using Cinnameg notation, we will write [: T :] for the type of a box that holds something of type T. Suppose that Bear $\leq$ Animal. How should [:Bear:] and [:Animal:] be related? Boxes have two fundamental operations: fetching and storing. If you fetch a value from a box of type [:Animal:], you do not necessarily get a Bear, so it should not be the case that [:Animal:] is a special case of [:Bear:]. But you cannot store a general Animal into a box of type [:Bear:], so, for the store operation, [:Bear:] is not a special case of [:Animal:] either. The most reasonable thing is to say that boxes are *invariant*; [: A :] $\leq$ [: B :] only when $A = B$. That prevents the type checker from allowing a box to be used in a context (either a fetch or a store) where it would not make sense.

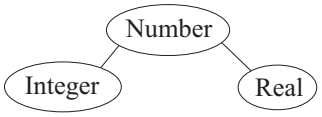
Java offers arrays and linked lists, through two separate mechanisms. Arrays were part of the first version of Java, and they are defined to be covariant. Type Integer[] (an array of integers in Java) is considered a subclass of Number[] (an array of numbers). That means a program can ask to store an object of x type Number[] into a variable v of type Integer[]. A type check is done a run time to ensure that x has type Integer[], and the program throws an exception if that is not the case. So you can use an array of integers where an array of numbers is required, but the price is a loss in static type checking.

Linked lists are provided through the Java generics facility, only added at Java 1.5, and all generics are considered invariant. So LinkedList<Integer> is not considered a subclass of LinkedList<Number>. The reason for the difference between linked lists and arrays is that the Java abstract machine knows about arrays but is unaware of the existence of generics, and so cannot insert the type check that would be required for linked lists.

20.6. Virtual methods

Suppose that you need to implement arithmetic on numbers with large numbers of digits. You create two classes, class Integer to handle large integers and class Real for real numbers with very high precision. For each class, you implement the operations sum, difference and product. Now you decide to implement some operations that can be defined in terms of sum, difference and product. For example, you define $\text{sqr}(x) = x^2 = x \cdot x$ using the product method.

If classes Real and Integer are implemented separately, then you must write the sqr method twice, once in class Real and once in class Integer. But the point of the class hierarchy is to give you places to write methods that are shared by several classes. A class Number seems to be called for, serving as a place to write methods that are shared by classes Integer and Real. So you create a class hierarchy as follows, where Integer and Real each have base class Number.



Looking at the methods in more detail, suppose the product method runs inside one number (the agent) and takes another number as a parameter, returning the product of the parameter with the agent number. The sqr method does not need any explicit parameters; it just takes the product of the agent number with itself. An initial stab at writing class Number looks like as follows, where self indicates the object that is running the method.

```
class Number
  instance methods:
    sqr() = product(self)
end class
```

But there is a serious difficulty with that; the sqr method refers to a product method that is not defined in class Number. Rather, product is defined in classes Integer and Real. Since an object of class Number has no product method, it cannot perform such a method.

An obvious thing to do is to define the product method in class Number. But that also meets with difficulties. The product method must be defined one way for real numbers and another way for integers. In fact, the product method has already been written in class Real and in class Integer. It would not make sense to attempt to implement it again in class Number. The point of class Number is to capture similarities of real numbers and integers, not to implement them again.

In a nutshell, the problem is that every kind of number needs to have methods sum, difference and product, so those methods *logically* belong to class Number. But those methods cannot be defined in class Number; they must be defined in subclasses of Number. Methods that have this characteristic are called *virtual* methods. Using the terminology of Chapter 13, normal (nonvirtual) polymorphic methods are defined parametrically, intended to work the same way in every subclass. Virtual methods are defined in an ad-hoc manner, with a separate definition for each (immediate) subclass. Classes Real and Integer do not inherit definitions of sum, difference and product from Number, but instead inherit a *responsibility* to define those methods themselves.

20.6.1. Abstract classes

A class that has at least one virtual method should have no instances, since those instances would not possess at least one of the required methods. Only its subclasses should have instances. Classes that do not possess instances are called *abstract classes*. Figure 20.5 shows a definition of abstract class Number. Think of it as a place to attach shared parametric method definitions (and occasionally shared variables).

20.6.2. Virtual methods and union types

Chapter 14 shows how to define a type with multiple constructors. For example, Figure 14.8 defines a type BinarySearchTree with two forms, an empty tree and a nonempty tree. Subclasses and virtual methods offer another way to achieve the same goal.

Imagine that you intend to create a module for symbolic computation on expressions, as software such as Mathematica does. For example, you would like a method that takes the derivative of an expression with respect to x. There are several kinds of expression: the independent variable, x; a constant, such as 1.0; a sum of two given expressions; a product of two expressions; etc. To handle that, you define class Expression and then add a subclass for each kind of expression. Since each kind of expression has its own rule for computing

```
abstract class Number
  virtual instance methods:
    sum:        Number -> Number
    difference: Number -> Number
    product:    Number -> Number
  instance methods:
    sqr() = product(self)
end class
```

Figure 20.5: **Class Number with virtual methods.** The sqr method can use product, even though product is not defined here. The class is abstract to indicate that it has at least one virtual method.

the derivative, the derivative method is virtual, and the methods for each kind of data are placed with that kind of data, in its class.

When a single type with variants is used instead, as done in Chapter 14, adding a new kind of expression requires modifying functions by adding a new case to each that handles the new kind of expression. But the object-oriented approach avoids that. Adding a new kind of expression just requires adding a new subclass of Expression and defining the virtual methods for that class. There is no need to modify any existing classes or methods at all.

Now imagine providing expressions and symbolic computation on them in a library. Keep in mind that most programmers are not allowed to modify the library, but can only use it. The single-type approach allows the writer of the library to include a fixed collection of expression forms, but does not allow anyone else to add new expressions. Classes and virtual methods allow anyone to add new kinds of expressions in his or her program without modifying the library at all.

Sometimes you do not want to add a new variation but to remove one because it is no longer needed. That is also easy to achieve using the object-oriented approach; you just remove the subclass that implements that variation (and any place where constructors of that subclass are used). There is no need to modify the implementations of other classes.

20.6.3. Virtual methods as function parameters

Some programming languages allow you to pass functions as parameters to other functions, but most object-oriented languages do not allow you to do that. But sometimes you can simulate function parameters by using virtual methods.

Suppose that you want a function or method, time(f), that runs another function f and tells you how long f takes to run. In an object-oriented programming language, you create a class called Timer, as in Figure 20.6, containing a virtual method called run that represents the function that is to be timed. It is a *hook* where the actual function can be attached. To time a function, you define a subclass of Timer where the run method runs the desired function. Instead of asking one object to time the execution of a method in another object, you let the subclass inherit the time method, so an object of the subclass knows how to time itself.

Chapter 10 showed that the ability to pass functions as parameters to other functions allows the creation of versatile libraries containing functions that implement entire classes of algorithms. Virtual methods in class libraries can be used to achieve similar generality.

```
abstract class Timer
  virtual instance methods:
    run: () -> ().
  instance methods:
    time() = t2 - t1 |
      {t1 = currentTime()}
      (run())
      {t2 = currentTime()}
end class
```

Figure 20.6: **Class Timer.** The function to be timed is virtual. For each function that you want to time, you create a subclass containing an appropriate definition of run.

20.7. Exercises

20.1. What is the difference between polymorphism and inheritance in object-oriented programming?

20.2. Suppose that you are writing in a statically typed object-oriented programming language. Suppose that B is a subclass of A.

(a) Should you be allowed to store an object of class B into a variable of nominal type A? Why or why not?

(b) Should you be allowed to store an object of class A into a variable of nominal type B? Why or why not?

20.3. What is one motivation for restricting a language to single inheritance?

20.4. Why is inheritance important to someone who only wants to use a class library, and does not intend to create his or her own class library?

20.5. Constructors are not inherited by subclasses. Why not?

20.6. One style of implementing a dispatch table for a single-inheritance language is to store, with each class, only a table of the methods defined in that class, along with the number of the first method defined in the subclass, and the number of the base class. For example, suppose that Bear is a subclass of Animal, class Animal contains four methods, and class Bear contains two additional methods. Then the dispatch table entry for Bear contains (1) the number [4] of methods that it inherits, (2) entries for the two methods defined in class Bear, and (3) the number of class Animal, where additional methods can be found.

(a) Write a pseudo-code algorithm to look up a method definition in the dispatch table. Assume that the i-th row of the table T contains information $T[i]$.baseclass (the number of the base class), $T[i]$.numInherited (the number of methods in the base class) and $T[i]$.localMethods (an array of methods for this class that are added). Lookup(c,n) should return method number n in class number c. Be careful to note that $T[i]$.localMethods[1] is the first method defined locally in class i, not the i-th method of class i.

(b) What is an advantage of this implementation of the dispatch table?

(c) What is a disadvantage?

20.7. A subclass typically adds methods to those that it inherits. Should it be allowed to remove methods, by choosing neither to inherit them nor to implement them? What problems might that cause?

20.8. Using the syntax of this chapter, write a definition of a class that is a subclass of Queue, but that also allows adding a string to the beginning of the queue instead of to the end. Include constructors.

20.9. C++ is an unusual object-oriented programming language in that, by default, it does not use the tag on an object to find a method in the dispatch table. Instead, it employs *static method selection*, where the class is the one associated with the type of the variable that holds the object. For example, if a variable m of class Animal contains an object b of class Bear, then m.roar() uses the the roar method in class Animal, not the one from class Bear. But if a method, such as roar, is declared *virtual*, then C++ uses the usual *dynamic method selection*, based on the tag of the object itself.

(a) We saw that an overriding method often wants to perform the method that it overrides. Argue that selection of that method is a kind of static selection, depending on the class where the overriding definition occurs, and should not be based on the tag of the object performing the overriding method.

(b) Give an example where static method selection will not work, and dynamic method selection is required.

(c) Explain how static method selection might lead to a more efficient program.

20.10. Why aren't objects that are tagged by abstract classes typically created?

20.11. Our abstract classes have no constructors. Can you think of a circumstance where an abstract class would need a constructor even though it is not possible to create an object whose tag is abstract class?

20.12. Does the dispatch table need a row for an abstract class? Explain your reasoning.

20.13. Assume that Bear $<$ Animal in a particular hierarchy. (We say that $A < B$ if $A \leq B$ and $A \neq B$.) Which of the following is true?

(a) Bear $\leq$ Bear

(b) (Bear,Bear) $\leq$ (Bear,Animal)

(c) (Bear,Animal) $\leq$ (Animal, Bear)

(d) (Bear $\rightarrow$ Bear) $\leq$ (Bear $\rightarrow$ Animal)

(e) (Bear $\rightarrow$ Bear) $\leq$ (Animal $\rightarrow$ Bear)

20.14. Extend the rule for relationships among ordered pairs to relationships among arbitrary tuples. When is $(A_1, \ldots, A_n) \leq (B_1, \ldots, B_n)$?

20.15. Implement binary search trees, with insertion and membership testing, by creating three classes representing (1) a general binary search tree, (2) an empty tree, and (3) a nonempty tree. Make the member and insert methods virtual, and implement them in the subclasses. Make the insert function modify the tree. For simplicity, assume that each value in the tree is a string. Now add a method that adds all of the members of a given list to the tree. Where should that method be written?

20.16. Write a definition of linked lists using three classes, (1) a class LinkedList, (2) a subclass EmptyList of LinkedList, and (3) a subclass NonemptyList of LinkedList that holds a head and a tail of a list. Create and implement virtual methods null(), which tells whether the list that runs it is empty; head(), which returns the head of a list; and tail(), which returns the tail of a list. Add constructors nil(), which produces an empty list; and cons(h, t), which produces a list whose head is h and whose tail is t. Finally, write a function length() in two different ways. First, write it as a function entirely in class LinkedList. Then rewrite it as a virtual method, with a separate implementation in each subclass.

20.17. After implementing linked lists as in the preceding problem, write a class Mapper that has a virtual method f and a nonvirtual method, map, that takes a linked list L and produces the result of mapping f onto L. For example, if z is an instance of a subclass of Mapper that defines f by $f(x) = x^2$, then z.map([2,4,6]) = [2^2, 3^2, 4^2] = [4,9,16].

20.18. (This problem requires some familiarity with C++ and access to a C++ compiler.) In C++, write a class A with a virtual method. In C++ you are allowed to give an implementation of a virtual method. Implement the virtual method so that it prints something, so that you can see that it is running.

Now use the virtual method from inside a constructor for class A.

Now write a subclass B of class A, and override the virtual method so that it prints something different. Make a constructor for class B that uses the constructor for class A. Now use the constructor for class B to build an object of class B. Which implementation of the virtual method do you think will be used from inside the constructor, the one written in class A or the one in class B? Try the program. Which implementation of the virtual method was actually used? Can you explain what is happening?

20.19. Do exercise 14.8, but instead of creating just one type Expression, create a class Expression with a subclass for each type of expression. Define the derivative and simplification functions. Where should the definition of derivatives for sum be written? Is there anything that can be written in Expression and shared in its subclasses?

20.8. Bibliographic notes

LaLonde and Pugh [63] discuss pragmatics of inheritance. Meyer describes Eiffel [69], an object-oriented language based on multiple inheritance. Virtual methods have been an important component of object-oriented languages since Simula [16]. The Timer class is similar to an example in Arnold, Gosling and Holmes [6].

Chapter 21

Smalltalk

21.1. Introduction to Smalltalk

Object-oriented programming got its start with the programming language Simula, which introduced classes and inheritance by adding them to Algol 60. Objects and classes were primarily intended to aid in performing simulations. The Simula library contains a class called Simulation that provides support for simulations, handling the mechanics of creating, scheduling and selecting events, with the idea that a programmer should create a subclass of Simulation for doing a particular simulation, and for deciding just which events to schedule.

The convenience and flexibility of class libraries soon became apparent, and they were especially attractive for the emerging area of graphical user interfaces, where large, versatile libraries are a necessity. But, rather than grafting classes onto an existing language, as Simula had done, it seemed sensible to make classes the central concept of a new language, suitable for programming in an environment that employed graphical interactions and new (at the time) tools, such as the mouse. Smalltalk was developed in that spirit to take full advantage of classes and objects. As one of the first truly object-oriented languages, Smalltalk serves as a prototype for object-oriented languages today.

Smalltalk went through several revisions over time; Smalltalk-80 is the first dialect to be widely used. This chapter describes the general ideas behind Smalltalk, including some of its interesting characteristics, rather than the details of one version. We do not attempt to describe the entire language, and only touch on the library.

21.2. Classes

Smalltalk supports single inheritance. That is, each class has just one base class. There is one class, Object, that has no base class; it is at the top of the class hierarchy.

Describing classes

One unusual feature of Smalltalk is that it has no standard syntax that programmers are supposed to use for describing classes.[1] There is a syntax for writing individual methods, but

[1] Smalltalk programs have exchange syntax that is used for moving programs from one machine to another, but programmers normally do not see it.

none for listing instance variables of a class, or for indicating relationships among classes, or whether methods are public or private. Instead, a Smalltalk implementation should come with a graphical editor that gives you menu options, such as creating a new class, adding a variable, adding a method, or indicating whether a method is public or private. We will write classes in a pseudo-syntax similar to that used in preceding chapters.

Smalltalk is typeless, so variables are just listed, by their names, and there is no indication of the types of parameters, or of the result from a function. By convention, parameter names are chosen to have an indication of the type that you expect them to have. For example, a parameter called aNumber is one that you expect to be a number. But there is nothing to prevent you from passing any object as the parameter called aNumber.

21.3. Lexical issues

Smalltalk is case-sensitive, so be careful not to confuse r with R. Smalltalk if free-form, so a line break is treated like a space in most places. Comments are written in double quotes.

Constants and identifiers

Smalltalk identifiers begin with a letter and contain letters and digits. Identifiers are used as the names of variables, methods and classes. Private variables in a class have names that begin with lower case letters, and names of public variables (much less common) begin with upper case letters.

Numeric constants are written in the usual way, such as 58 or 91.4. Character constants are written preceded by a dollar sign. For example $X is the upper case X character, and $+ is the character +. Strings are written in single quotes, such as 'Smalltalk'. Smalltalk also supports symbols, similar to the symbols of other languages, such as Scheme. You write a symbol by preceding it by a # sign. Constant #lion is the symbol whose name is "lion".

Every object has a class, and classes have names that start with upper case letters. Constant 41.2, for example, has class Float, $A has class Character, 'a string' has class String and #yak has class Symbol. Integers are somewhat complicated because there are three classes, SmallInteger, LargePositiveInteger and LargeNegativeInteger, for integers. Each is a subclass of abstract class Integer. Constant 35 has class SmallInteger.

There is an object called **nil**, of class UndefinedObject, that is the default initial value of variables.

21.4. Expressions: using methods

All computation is performed by methods. There are three different syntactic forms for methods.

1. A *unary* method has an identifier as its name, and is written immediately *after* the object that performs it. For example, expression (x isNil) returns true if x is nil, and (x reciprocal) returns $1/x$ when x is a number.

2. A *binary* method has a name that is one or two special characters, or just a comma, and is written between the object that runs it and its parameter. For example, ($x + 5$) runs method + in object x with parameter 5, and (x <= 28) runs method <= in object x with parameter 28. Special characters are the members of string "`+/\*~<>=@%|&?!`".

Even fundamental operations such as addition and multiplication of numbers are accomplished by methods running in objects. Expression 3 * 4 asks object 3 (a number object) to run method * with parameter 4; it yields result 12, an object of class SmallInteger.

There is only one level of precedence among binary operators, and all operators are left-associative. So expression $3 + 2 * 5$ yields 25; it is computed by first adding 3 + 2, and multiplying the result by 5. Use parentheses to control the parsing. Unary methods have higher precedence than binary methods.

3. A *keyword* method takes one or more parameters, each preceded by a keyword. Each keyword ends with a colon, and the name of the method is formed by concatenating the names of the keywords. For example, expression

```
frog jump: 2 over: toad
```

selects a method called jump:over: from object frog, and runs it with parameters 2 and toad. Keyword messages have lowest precedence.

The Smalltalk library provides a large collection of methods. Numbers support the binary methods +, −, ∗ (multiplication), / (division), // (integer division, throwing away the remainder) and \\(integer remainder, or modulus). Division of integers can yield objects of class Fraction, which are stored as fractions. So expression 1/3 returns the number 1/3, without rounding.

If s is a string, then (s at: n) is the n-th character of s, and (s at: n put: c) modifies string object s, replacing its n-th character by c. Expression (s, t) yields the concatenation of strings s and t, a new string object. For example, ('choco', 'late') yields string 'chocolate'. Parentheses are not required; they are only shown to set off the expressions.

21.5. Defining methods

A method definition begins with a heading that looks like a method call, with the object that performs the method missing. After that is a sequence of statements, separated by periods. (So a period is a separator, not a statement ender.) Each statement in a method definition has one of the following forms.

1. An expression is also a statement. It is evaluated, and its result is ignored.

2. An *assignment* has the form *variable* := *expression*, though some Smalltalk dialects use ← in place of :=.

3. A return statement has the form ^*expression*. It evaluates the expression and returns its value as the result of the method. Some version of Smalltalk use ↑ in place of ^.

For example, the following two method definitions define a keyword method called add:, with one parameter (called aNumber), and a unary method called jump. They assume that there is an instance variable called amount in the class.

```
add: aNumber
  amount := amount + aNumber

jump
  amount := amount - 1.
  ^amount
```

Self

A method refers to the object that contains it as **self**. You refer to variables that belong to self by their names, but to run a method that is part of self, you must mention self explicitly. For example,

```
leap
  self jump.
  self jump
```

defines unary method leap, which jumps twice. If a method does not return any value, as leap does not, then by default it returns self.

Cascading

If you want to ask one object to perform several methods in a row, you can *cascade* methods using a semicolon. Think of a semicolon as standing for the object that performed the preceding step. So the leap method can be written as follows.

```
leap
  self jump; jump
```

Local variables

A method can have local variables, for use only while running that method. At the beginning of the method, write the names of the local variables inside |...|, separated by spaces. For example, the following method returns aNumber to the eighth power.

```
eighthPower: aNumber
  |aNumberSq aNumberFour|
  aNumberSq := aNumber * aNumber.
  aNumberFour := aNumberSq * aNumberSq.
  ^aNumberFour * aNumberFour
```

Method selection

Smalltalk relies on a dispatch table, looking up a method based on the tag of the object that is asked to perform the method. There is an exception, however. To allow an overriding method to get the method that it is overriding, pseudo-object **super** is the same as self, but the method is selected from the row for the superclass of the current class, regardless of the tag on the object. A typical method that is overridden is one that initializes an object.

```
initialize: anObject
  super initialize: anObject.
  x = 0
```

initializes variables from the base class and then initializes x, which might be a new variable.

21.6. Blocks

Early versions of Smalltalk were quite functional in appearance, and methods could be defined by a set of (semantic, if not syntactic) equations. For example, a definition of factorial in Smalltalk-71 looks like this.

to 'factorial' 0 **is** 1
to 'factorial' :n **do** 'n*factorial n-1'

Later versions of Smalltalk opted for a more imperative feel, but the functional heritage is still present in Smalltalk's first-class functions, called *blocks*. A block is a function that is written in-line inside a method. To create a block with no parameters, enclose a group of statements inside square brackets. For example, statement

```
blk := [count := count + 1.
        size := size - 1]
```

puts a block into variable blk that, when run, adds 1 to count and subtracts 1 from size. A block is an object, and, to run it, you ask it to perform unary method **value**. So

```
blk value
```

has the effect of adding 1 to count and subtracting 1 from size. A block produces a value, the value of its last expression. So

```
x := 5.
aBlock := [x + 1].
x := 9.
val := aBlock value
```

sets variable val to 10, the value of expression $x+1$ at the time when the block is evaluated. There is an empty block, [ ], whose value is nil.

A block can have parameters, just as other functions can. A block with one parameter called x has the form [:x | ...], and a block with two parameters x and y has the form [:x :y | ...]. In general, each parameter is preceded by a colon, and the list of parameters is ended with a vertical bar. To evaluate a block with one parameter, use keyword method **value:**, with the parameter. A block with two parameters uses method **value:value:**, and a block with n parameters uses method **value:value:...value:**, with n parameters passed to it. For example,

```
z := [:x :y | x + (y * y)] value: 3 value: 5
```

sets z to 28.

Blocks are not truly autonomous functions. If, during evaluation, a block performs a return statement ($\hat{}E$) then it causes the method that evaluates the block to return. So

```
x := [^1] value.
y := 0
```

will not set x or y to a value. It will return value 1 from the method when it evaluates the block.

21.7. Boolean objects and comparisons

Smalltalk provides a standard class, Boolean, with two subclasses, True and False. Class True has just one instance, **true**, and class False has just one instance, **false**. Methods that produce Boolean results include comparisons of numbers (=, ~=, <, <=, >, >=).

You can compare any two objects to see whether they are the same object. Expression $x == y$ yields true if x and y are the same object, and $x \sim\sim y$ yields true when x and y are different objects. Alternatively, you can use = and ~=, but the meanings are different. The idea of = is that $x = y$ should be true if objects x and y represent the same information, and each class typically decides how to implement that equality test. For example, two strings are equal under = if they contain the same characters, in the same order, but they are equal under == if they are exactly the same object (stored at the same place in memory).

21.8. Conditionals and loops

Smalltalk does not have any built-in control structures, such as conditionals and loops. The philosophy is to make the language small and simple, and to do as much with objects and classes as possible, pushing the language complexity and versatility into the library. Conditional expressions and loops are handled as methods of the Boolean class and of blocks.

If a is an object of class Boolean and b and c are two blocks, then (a ifTrue: b ifFalse: c) produces (b value) if a is true, and produces (c value) if a is false. For example, the Integer class provides unary method **factorial**, so that

```
x := y factorial
```

sets x to the factorial of y. The following is an implementation of the factorial method. Since it belongs to the Integer class, self is an integer.

```
factorial
  self = 0
    ifTrue: [^1]
    ifFalse: (self > 0 ifTrue: [^self * (self - 1) factorial]
                       ifFalse: [self error:
                                 'factorial of a negative number']
```

You can also use method ifTrue: (which does nothing if the object is false) and ifFalse: (which does nothing if the object is true). An alternative implementation of factorial is as follows.

```
factorial
  self = 0 ifTrue: [^1].
  self > 0 ifTrue: [^self * (self - 1) factorial].
  self error:  'factorial of negative number'
```

It is important that the arguments of ifTrue: and ifFalse: are blocks, for two reasons. First, you do not want to run them before you even perform the test; they should be deferred, as blocks are. Second, the ifTrue:ifFalse: method will ask one of those parameters to perform method **value**, as blocks do.

You can create a loop using method whileTrue: on a block object. If b and c are blocks, then (b whileTrue: c) evaluates block b, which should yield a Boolean result. If b

yields value true, then block c is evaluated, and b is evaluated again. Evaluation continues until b yields false. When the loop is done, it returns nil. For example, here is another implementation of factorial, this time using a loop. Remember that self is the integer whose factorial you are computing.

```
factorial
  |n r|
  n := self.
  r := 1.
  [n > 1] whileTrue: [
    r := r * n.
    n := n - 1].
  ^r
```

You can ask an integer n to evaluate a block n times. For example,

```
5 timesRepeat: [n := n * n]
```

squares n five times. Integers can also create counting loops. Statement

```
1 to: 10 do: [:i | count := count + i]
```

performs the indicated block for $i = 1, \ldots, 10$.

21.9. Constructors

Among other things, a class is an object factory, and it performs methods that create new instances of itself. Constructors are class methods, and each new class is automatically given a constructor called **new** (or **new:** — see indexed variables below) that creates a new, uninitialized member of the class. For example, expression `Widget new` asks the Widget class, as an object, to run its new method, which returns a new instance of class Widget. You can create other constructors by any names that you like. For example, if instances of class Point are points in the plane, then you might create a constructor called x:y: that creates a new point with a given x- and y-coordinate, as follows. It is a class method of class Point, so self is class Point, and expression (self new) creates a new, uninitialized Point.

```
x: x y: y
  p := self new.
  p initializex: x y: y.
  ^p
```

Instance method initializex:y: of class point is as follows, assuming that class Point has two instance variables x and y.

```
initializex: xNumber y: yNumber
  x := xNumber.
  y := yNumber
```

Now, the following creates a new initialized point called p.

```
p := Point x: 0 y: 2
```

Notice that this uses Point as an object, asked to run method x:y:.

21.10. Arrays

Constant arrays

Smalltalk supports lists, which it calls *arrays*. The list that would be written [1, 2, 3] in Cinnameg is written #(1 2 3) in Smalltalk. Since Smalltalk is typeless, you can mix different kinds of things in an array, and there is no distinction between a list and a tuple. Smalltalk value #(1 $X) corresponds to the ordered pair (1, 'X') in Cinnameg. You can only write an array of constants in this syntax. If x is a variable, #(x) does not give you an array that contains the value of variable x, but instead gives you a singleton array holding the symbol #x.

To create a list of lists, you only write the # sign once. For example, Smalltalk's #((1 'one') (2 'two')) is similar to the constant [(1, "one"), (2, "two")] in Cinnameg. To write a symbol in an array, you do not precede it by #. So #(one two three) is an array of three symbols, #one, #two and #three. The empty array is #( ).

Creating and working with arrays

You can create an array of a given size n, filled with nil values, using expression (Array new: n). For example,

```
a := Array new: 10
```

creates a new array of size 10. Operations on arrays include the following. In all cases, x is an array. Parentheses are shown to set off expressions, but are not required.

1. (x at: n) is the n-th value in x, numbering from 1. Arrays are objects, and can be changed. Statement (x at: n put: v) modifies array x so that (x at: n) yields v.

2. (x first) yields the first member of x, and (x removeFirst) returns a new object, the array produced by removing the first member of x. So first and removeFirst are similar to the Cinnameg head and tail operations.

3. (x addFirst: v) produces a new array object obtained by adding v to the front of array x. AddFirst is similar to the Cinnameg :: operator. For example, if x is array #(1 2 3) then (x addFirst: 0) yields a new array holding #(0 1 2 3). Similarly, (x addLast: v) produces a new array object that looks like x, but with v added to the end.

4. (x size) is the current size of array x.

5. The comma operator concatenates two arrays. So (x, y) is a new array object that holds all of the members of array x followed by all of the members of array y.

6. (x do: b) takes a block b with one parameter. It evaluates (b value: v) for each value v in array x. Use this to loop over the members of an array, where block b is the body of the loop. It is similar to map in Cinnameg.

For example,

```
a := Array new: 20.
1 to: 20 do: [:i | a at: i put: 0]
```

creates an array of 20 values, all 0.

Indexed variables

A major design goal of Smalltalk is to provide a relatively small number of primitive concepts and to let all other ideas be written in Smalltalk. The array class, for example, is written in Smalltalk, as part of the library. But then, where does the basic representation of arrays and indexing come from?

Smalltalk allows a class to have a single implicit array, called an indexed variable, accessed by an index instead of by name, using at: and at:put: methods. A class that has an implicit array uses constructor new: rather than new, where the parameter tells the size of the array. The array class is simply defined to have an indexed variable. Remaining methods, such as do:, are written in Smalltalk.

21.11. An example class

Figure 21.1 shows a class of queues, similar to the queues in Chapter 19. Recall that Smalltalk has no standard syntax for describing classes, only for methods.

21.12. Overriding and virtual methods

When you create a subclass of another class, you can override methods, as discussed in Chapter 20. Smalltalk has no special support for virtual methods or abstract classes. Instead, if a method cannot be implemented in a given class, then it is defined to yield an error, and should be overridden in subclasses. For example, if method find: cannot be implemented in class Collect, then it is written as follows.

```
find: aValue
  self error: 'find not implemented for class Collect'
```

An abstract class is any class that contains a method that must be overridden in subclasses.

21.13. Classes as objects

When you run a constructor or other class method, you think of the class an an object that performs the method. So classes are objects. But in Smalltalk, every object has a class, so every class must itself belong to a class. Think of a class as an object factory. The Queue class cannot belong to class Queue, since a queue factory is not a queue any more than a bicycle factory is a bicycle.

Metaclasses

The solution adopted by Smalltalk is to create new classes, called *metaclasses*, whose instances are other classes. In fact, for every class C, there is a metaclass called C *class*, whose only instance is class C. For example, class *Integer class* has just one instance, class *Integer*. The metaclasses are related to one another in the same way as their instances. For example, *SmallInteger* is a subclass of *Integer*, so *SmallInteger class* is a subclass of *Integer class*. At the top of the hierarchy of metaclasses is a class called *Class*. All metaclasses are subclasses of *Class*.

```
Class Queue
  instance variables:
    "members is an array of the members of the
    queue, from front to back"
    members
  class methods:
    new
      q := self basicNew.
      q initialize
  instance methods:
    insert: anObject
      members addLast: anObject
    delete
      |result|
      result := members first.
      members := members removeFirst.
      ^result
    makeEmpty
      members := Array new: 0
    isEmpty
      ^(members size = 0)
  private instance methods:
    initialize
      self makeEmpty
End class
```

Figure 21.1: **A Queue class, with methods written in Smalltalk.** This class overrides new. Standard class method **basicNew** is the initially the same as **new**, and it can be used when **new** is overridden.

Metaclasses are used for organizing the Smalltalk class editor. The variables and methods listed under the metaclass are class variables and class methods, as described in Chapter 19. They can be used in instance methods or in other class methods.

Getting the class of an object

Every object responds to unary method **class** with its class. So

```
c := 3 class.
```

sets $c =$ `SmallInteger`. Suppose that, inside class Widget, you try to make another object of the same class as self. You might try the following.

```
cpy := Widget new
```

But that creates a new object of class Widget. Remember that methods should be polymorphic, and you while writing the methods of class Widget, you should think about subclasses that might be added later. A more sensible polymorphic method would work as follows.

```
cpy := self class new
```

Now, regardless of the actual class of self, cpy has the same class, and this method is suitable not just for Widget but for any subclass of Widget.

The class of a metaclass

If all classes are objects, and every objects has a class, then even the metaclass `Integer class` must have a class. Every metaclass is an instance of class MetaClass. All of this yields a fairly complex hierarchy, and, in truth, there is more to metaclasses than we have space to go into here. But the key point is that Smalltalk, by its organization of classes and metaclasses, sticks to the principle of object-oriented programming that every object has a class, and simultaneously allows you to think of a class as an object that performs class methods, such as constructors.

Introspection

Smalltalk allows a program to do much more than just get the class of an object dynamically. A program can inspect the class hierarchy by moving up and down (to superclasses and subclasses), getting metaclasses, and finding methods and variables. It can even add new methods and variables to a class or remove methods and variables while the program is running. A program can write code, as a string, compile it on the fly, and add it to a class. That allows the Smalltalk development environment itself to be written in Smalltalk. Just one example of the methods that are available is sourceCodeAt:, where, for example, expression

```
Array sourceCodeAt: #do:
```

produces a string that is the source code of the do: method in class Array.

21.14. Exercises

21.1. Is Smalltalk intended to be a large or a small language? How does it provide a great deal of functionality to the programmer?

21.2. What do you think is the purpose of having two subclasses, True and False, of Boolean, rather than just having two instances, true and false, of class Boolean. Think about how operations such as ifTrue:ifFalse: are implemented. Does having two subclasses simplify their implementation?

21.3. Are there any dangers in allowing a program to make modifications to classes on-the-fly? Explain what might happen.

21.4. Smalltalk allows most methods to be overridden in subclasses, but encourages you not to override some of them. Explain difficulties that can happen if you override some library methods.

21.5. Blocks have class BlockContext. Write a definition of the whileTrue: method of the BlockContext class. You should not need to use a loop to do this. Smalltalk allows method definitions to be recursive.

21.6. Section 8.2.4 discusses tail recursion. Based on your answer to question 21.5, do you think that the Smalltalk system performs efficient tail recursion? Why or why not?

21.7. Write an implementation of method quadraticWithCoefficients: a and: b and: c. It should be part of the Float class, and should return array (y,s) where s is an array of the real solutions of equation $ax^2 + bx + c = 0$ and y is the value of expression $ax^2 + bx + c$ evaluated at x = self.

21.8. Most object-oriented systems do not allow you to modify the classes in the library, but instead encourage you to create subclasses of them. Based on what you have read in this chapter, do you thing that a Smalltalk system allows you to add a new method to library class Float? Why or why not?

21.9. Write an implementation of stacks in Smalltalk as a class Stack.

21.10. For question 21.9, why is it not necessary to write a parameterized class Stack(T)?

21.11. A block normally does not outlive the method that created it. Can you see how a block might outlive its method? Can you see a problem with that? Explain.

21.15. Bibliographic notes

Texts on Smalltalk include Goldberg and Robson [39], Budd [18], LaLonde and Pugh [61, 62], Shafer, Herndon and Rozier [88], and Smith [90]. The journal, *The Smalltalk Report*, contains numerous articles on Smalltalk and how to use it. Alan Kay [58] gives a history of the development of Smalltalk. The Smalltalk-71 factorial method definition is from Kay's article.

Chapter 22

Semantics

22.1. Introduction to semantics

The semantics of a program tells you exactly what that program does for every possible input. The semantics of a programming language tells you what every syntactically well-formed program does in every possible situation in which it might run. That involves describing every construction and every basic word or action in the language. Semantics is a major part of a programming language description, and a substantial part of what you read in this book concerns it. This chapter only covers broad notions in semantics, with later chapters providing more detail.

22.1.1. Operational semantics

To describe what an imperative program does when it runs, you explain the sequence of changes that it makes. A program starts in a particular *state*, indicating the initial values of variables, and what is in the input. Each basic step of the program produces another state. The overall meaning of the program, or its *operational semantics*, is the sequence of states that it goes through. Chapter 22 looks at methods for carefully describing operational semantics.

22.1.2. Denotational semantics

Instead of saying what an expression *does*, the semantics of an expression can rely on saying what it computes, or what it *denotes*. For example, expression $3 + 4$ denotes 7 since that is the value that it has when it is computed. A *denotational semantics* of a declarative programming language tells what every syntactically correct expression denotes, and suppresses details about how the result is computed.

There is still a notion of state, giving values of variables. For example, in order to obtain the value of expression $y + 2$, you need to know the value of y. But, in a denotational semantics, there is no notion of one state coming after another in the computation. A state just provides static information about variable values. The denotational semantics of an expression is given by a function that takes a state and produces the value of the expression in that state. Chapter 23 discusses denotional semantics.

22.1.3. Axiomatic semantics

Instead of saying how a program works or what it computes, *axiomatic semantics* concentrates on what a programmer can know about what a program does or computes. An axiomatic semantics provides a collection of rules for proving that given programs have given characteristics. Chapter 24 gives an example of an axiomatic semantics.

22.1.4. Partial semantics

Programming language designers sometimes prefer to provide only partial information, in order to give language implementors some flexibility. A common example is computation of pairs, triples, etc. If A and B are expressions then the ordered pair (A, B) can be computed in two obvious ways. The implementation might

1. Compute the value a of expression A.
2. Compute the value b of expression B.
3. Build the ordered pair (a, b).

or it might instead

1. Compute the value b of expression B.
2. Compute the value a of expression A.
3. Build the ordered pair (a, b).

For most programs it does not matter which order is chosen, but sometimes the order influences how efficiently the computation can be done, so language designers prefer to leave the choice of order up to the implementor. The language definition simply does not say which order is used.

22.1.5. Relational semantics

In some languages an expression can produce more than one result. That is not the same as leaving the result up to the language implementor, nor the same as producing an ordered pair as its result; instead, you say that an expression might produce 1, and might produce 2, possibly by backtracking. The language implementation explores *all* of the potential results.

A variation on denotational semantics can be used for those languages. Instead of being a function from states to values, the semantics or an expression is a *relation* between a state and a value. A relation is a set of pairs. Imagine an "expression" betw(x, y) that produces any integer (and all integers) between x and y, including x and y. Let s be the state $\{x = 4, y = 6\}$. Then the relation that is the semantics of betw(x, y) contains $\{(s, 4), (s, 5), (s, 6)\}$ as well as more pairs for other states.

22.1.6. Semantic black holes and infinite loops

In most programming languages it is possible to write a program, or part of a program, that never stops, but keeps computing forever without producing any result. For an operational semantics, that means the program goes through an infinitely long sequence of states. A denotational semantics has a more difficult problem. Recall that a denotational semantics is supposed to tell the value of every expression in every state. But an expression simply does not have a value when its computation never stops. So there does not appear to be

any answer that a denotational semantics can attach to some expressions. The solution is to attach a special *infinite loop* meaning to an expression whose evaluation does not stop.

An infinite loop affects not only one statement or expression, but everything after it, since any action that should have been done after the nonterminating computation will never be performed. The program appears to enter a black hole from which it can never emerge. That significantly complicates the semantics, since every semantic rule must take infinite loops into account. Even the rule for adding two numbers needs to say what the answer is when one of the operands is an infinite loop. (The sum is also an infinite loop.) Some languages, such as Charity, do not allow a programmer to write a program that never terminates. The semantics of Charity is significantly simplified by that.[1]

There is another kind of black hole that does not involve an infinite loop, and that is often more difficult to deal with for a programmer. A few programming languages allow well-formed programs whose semantics are so difficult to understand that the language designers are not able to give any meaning to them at all. Once such a computation is performed, not only is the result of that part of the program unspecified, but everything that the program does from then on is undefined. For example, the semantics of C indicates that, if x is a real number and you perform statement `x = 3.5` then afterwards, variable x will have value 3.5. But if you write

```
a[n] = 0;
x = 3.5;
```

and n is not in the range of indices supported by array a, then there is no guarantee that x will hold 3.5 afterwards. It is possible that the program stores the constant 3.5 into memory somewhere and performs statement $x = 3.5$ by copying from that memory location into variable x. C programs do not check array indices; $a[n]$ is just the memory location n spots from the beginning of array a in memory, and if it happens that $a[n]$ refers to the memory location where 3.5 is stored, then that value might be overwritten.

Obviously, you want to avoid semantic black holes. With the exception of infinite loops, most language definitions have no black holes at all. C and C++ are examples of languages whose semantics are replete with them. The committee that wrote the definition of the ANSI C++ standard seemed to like them so much that it decided to add some new semantic black holes to the language. Expression `++x` has a side-effect of adding one to variable x, and has the new value of x as its value. Statement `x = ++x` appears to add 1 to x and then to store the result into x again, causing a redundant operation to be performed. But according to the ANSI C++ standard, performing statement `x = ++x` causes a program to enter a semantic black hole! Almost anything could happen.

22.1.7. Resource limitations and semantics

Every computer has resource limitations. A memory limitation might cause a computation to fail when it tries to allocate memory, even though the language semantics says that the computation should succeed. Generally, resource limitations are not explicitly given as part of the semantics of a programming language since the available resources vary from

[1] Unfortunately, there is a curious and subtle result of computability theory: For every (implementable) programming language L where all programs must terminate, there exists a problem that can be solved in a general programming language, by a program that always terminates, but that cannot be solved by any program written in language L. So Charity cannot solve all problems that general purpose languages can, even when you require programs in the general language to terminate on all inputs. Somehow, the ability to loop forever is critical, even when you do not use it!

one computer to another. The semantics of Cinnameg, for example, allows you to compute with arbitrarily large integers, and the language semantics does not put a limitation, such as 1,000,000 digits, on the size of integers. Any particular machine will necessarily impose some limits. When you encounter an error during computation caused by a memory limitation, the language implementation is not really considered to be broken, since there is nothing that it can do about a lack of memory.

22.2. Operational semantics

Semantic rules for an imperative programming language tell what each statement does and how the values of expressions are determined. This section examines a way of describing the semantics of imperative languages. A careful description of semantics is necessarily detailed and mathematical. You derive the semantics of a program by proving a fact, and parts of this chapter employ concepts from mathematical logic and proofs.

Semantic descriptions of entire programming languages tend to be large and to take time to understand, and languages whose descriptions are too mathematical tend to fair poorly because few programmers are willing to wade through that level of detail.[2] But careful semantic rules, particularly for portions of languages that are difficult to understand, can not only help programmers to understand those features, but can be especially useful to programming language implementors, who need to know exactly how implementations are required to behave.

22.2.1. States

The *state* of a computer at a moment in time tells the entire content of the computer's memory and everything else associated with the computer, such as the contents of files. (A full description of the state for an interactive program also needs to describe the state of mind of the program's user, so that you know what the user will do; otherwise you cannot say what the program will do. But to avoid that complication, we will restrict attention to programs that do not interact with a human.)

22.2.2. Atomic steps

A program runs by performing a sequence of fundamental, or *atomic*, steps, each step making a small change to the state. For example, a step might consist of adding two numbers and storing the result into a variable. More complex statements can be performed by doing a sequence of atomic steps. For example, assignment statement $w := x + y + z$ needs to add x and y, then add that total to z before changing w.

Part of a semantics of an imperative language is identification of the atomic steps that a program can make. Any mathematical semantics is only an approximation of what really happens in a given computer. One semantic description of a language might say that assignment statement $k := k + 1$ is an atomic action, taking just one step to perform. Another semantics of the same language might offer more detail, and say that statement $k := k + 1$ requires fetching k into a register, adding 1 to that register, and then storing the

[2]The report on the programming language Algol 68, for example, employs a lengthy and mathematical semantic description. Algol 68 is a large and complex language, and the fact that its report was so difficult to read was one factor contributing to Algol 68 not enjoying as widespread use as its predecessor, Algol 60.

register into k, for a total of three steps. Both take you to the same state at the end, but one takes a longer route there.

There is no obvious "right" choice of atomic steps. It would not be a good idea, for example, to model the semantics on the exact actions of one brand of processor, since then the language is inherently tied to that one processor. Our approach will be to choose actions that make the semantics easy to describe. Any set of atomic actions that, at a minimum, gives the correct final state of a program can at least be used to determine the answer that a program produces, and offers at least some idea of the steps that got it there.

22.2.3. Sequences of states

A program performs a sequence of atomic steps. But you can describe a sequence of actions implicitly by giving the states that result from the actions. For example, if the program starts in state s_1 and then moves to state s_2, you know that it performed some action that would change s_1 into s_2, and it is not necessary to say just what the action is. A semantics that is based on sequences of states is called an *operational semantics*.

22.2.4. A notation for states

For simplicity, we will only model variables, not files and other things connected to a computer. So a state tells the current values of all of a program's variables, or at least of those that have values. We assume that each variable has a name. We write variable values in braces; $\{x = 4, y = 7\}$ indicates that x has value 4 and y has value 7. A collection of variable bindings in braces is called a *simple state*.

An assignment statement changes the state. For example, performing assignment $x := x + y$, starting in simple state $\{x = 4, y = 7\}$, yields simple state $\{x = 11, y = 7\}$ (possibly after a few intermediate states).

If S is a simple state, write $\theta_S(x)$ for the value of variable x in simple state S. For example suppose that state W is $\{x = 4, y = 9\}$. Then $\theta_W(x) = 4$ and $\theta_W(y) = 9$.

Write $S[x := v]$ for the simple state that is obtained from S by either adding a new binding $x = v$ (if S has no binding for x) or by replacing the binding of x by v (if S has a binding for x). For example, if simple state W is $\{x = 4, y = 9\}$, then $W[x := 33]$ is simple state $\{x = 33, y = 9\}$, and $W[z := 5]$ is $\{x = 4, y = 9, z = 5\}$.

Full states

Simple states are a start, but they are not enough for describing semantics. For each state, an operational semantics needs to say which state comes next in a computation. But that depends on the program itself, and on where the program's thread of control is currently pointing, so that you know which statement the program will perform next. So the program needs to be part of the state.

A *full state* has the form $\langle P \parallel S \rangle$, where P is a program and S is a simple state. It is assumed that P starts with its first statement. For example, full state $\langle y := 3;\ z := 1;\ \parallel \{x = 0, y = 1\} \rangle$ indicates that the program has already reached a state where $x = 0$ and $y = 1$, and it is about to continue with statement $y := 3$, which will in turn be followed by statement $z := 1$.

Our semantics needs to indicate how expressions are evaluated, so a full state can contain an expression instead of a statement. For example, full state $\langle x + 5 \parallel \{x = 1\} \rangle$

indicates that expression $x + 5$ is being computed, and that x has value 1. The next full state could be $\langle 1 + 5 \parallel \{x = 1\}\rangle$, where the value of x has been fetched and substituted into the expression. Assuming that addition of integers is an atomic step, the next full state would be $\langle 6 \parallel \{x = 1\}\rangle$, where the expression is just a number.

22.2.5. The single-step relation

If F and G are full states, then we write $F \to G$ to indicate that G immediately follows F in a computation. That is, a single atomic step has been performed, converting full state F to full state G. For example, $\langle y := 3; z := 1; \parallel \{x = 0, y = 1\}\rangle \to \langle z := 1; \parallel \{x = 0, y = 3\}\rangle$. Notice that statement $y := 3$, having been performed, is removed, making $z := 1$ the next statement to be done.

The single-step relation also applies to evaluation of expressions. For example, $\langle x + 5 \parallel \{x = 1\}\rangle \to \langle 1 + 5 \parallel \{x = 1\}\rangle$ shows one step in evaluation of expression $x+5$, moving it closer to its final value. Some changes in a program do not remove a statement, but modify an expression within the statement. For example, assuming additions are performed from left to right,

$$\begin{array}{lll} & \langle x := 3 + 5 + y; z := 41; & \parallel \quad \{y = 2\}\rangle \\ \to & \langle x := 8 + y; z := 41; & \parallel \quad \{y = 2\}\rangle \\ \to & \langle x := 8 + 2; z := 41; & \parallel \quad \{y = 2\}\rangle \\ \to & \langle x := 10; z := 41; & \parallel \quad \{y = 2\}\rangle \end{array}$$

and evaluation continues from there. A sequence of full states $s_1 \to s_2 \to s_3 \to \cdots \to s_n$ is called an *evaluation sequence*. Figure 22.1 shows two short evaluation sequences, and Figure 22.2 shows a longer example involving a while loop.

22.2.6. A notation for describing an operational semantics

An operational semantics needs to define the single-step relation $\to$, where $S \to T$ indicates that full state S is immediately followed by full state T during computation. One way to define the $\to$ relation is via *rules of inference*, allowing you to draw conclusions, possibly based on other conclusions that you have already made. If Γ is a list of relationships and A is a relationship, then we write

$$\frac{\Gamma}{A}$$

to indicate that relationship A is true whenever all of the relationships in list Γ are true. For example, inference rule

$$\frac{\langle \alpha \parallel S\rangle \to \langle \beta \parallel T\rangle}{\langle v := \alpha;\ L \parallel S\rangle \to \langle v := \beta;\ L \parallel T\rangle} \tag{22.2.1}$$

tells a property of assignment statements, of the form $v := \alpha$. In this rule, v stands for an arbitrary variable, α and β stand for expressions, and L is any sequence of statements. Rule (22.2.1) says that, if you have already concluded that performing one step in the evaluation of expression α (in simple state S) yields expression β (and, possibly due to side-effects of the evaluation, new simple state T), then you can perform one step in the evaluation of assignment statement $v := \alpha$ (also in simple state S) by changing this assignment statement to $v := \beta$ (and changing the simple state to T). For example, suppose that you already know that $\langle 1 + 4 \parallel S\rangle \to \langle 5 \parallel S\rangle$ for any simple state S. Then you can use Rule (22.2.1) to infer $\langle x := 1 + 4;\ y := 9; \parallel \{z = 2\}\rangle \to \langle x := 5;\ y := 9; \parallel \{z = 2\}\rangle$.

$$
\begin{array}{rll}
 & \langle x := 2; y := x + 6; & \| \quad \{\}\rangle \\
\rightarrow & \langle y := x + 6; & \| \quad \{x = 2\}\rangle \\
\rightarrow & \langle y := 2 + 6; & \| \quad \{x = 2\}\rangle \\
\rightarrow & \langle y := 8; & \| \quad \{x = 2\}\rangle \\
\rightarrow & \langle & \| \quad \{x = 2, y = 8\}\rangle
\end{array}
$$

(a)

$$
\begin{array}{ll}
\langle x := 2; \textbf{ if } x > 4 \textbf{ then } y := x; \textbf{ else } y := 0; \textbf{ endif} & \| \ \{\}\rangle \\
\rightarrow \langle \textbf{if } x > 4 \textbf{ then } y := x; \textbf{ else } y := 0; \textbf{ endif} & \| \ \{x = 2\}\rangle \\
\rightarrow \langle \textbf{if } 2 > 4 \textbf{ then } y := x; \textbf{ else } y := 0; \textbf{ endif} & \| \ \{x = 2\}\rangle \\
\rightarrow \langle \textbf{if } \text{false } \textbf{then } y := x; \textbf{ else } y := 0; \textbf{ endif} & \| \ \{x = 2\}\rangle \\
\rightarrow \langle y := 0; & \| \ \{x = 2\}\rangle \\
\rightarrow \langle & \| \ \{x = 2, y = 0\}\rangle
\end{array}
$$

(b)

Figure 22.1: **Two evaluation sequences.** Each line represents one small step in evaluating a program. State $\langle \text{P} \parallel S\rangle$ indicates that we are about to run program P, where the current variable values are given by simple state S. Computation stops when the program is empty. These evaluation sequences follow the general rules in Figures 22.3 and 22.4.

Some rules do not require that you already know anything. We write

$$\frac{}{A}$$

to indicate that A is always true, and does not depend on any previously shown facts. For example, a basic fact for variables (as a simple form of expression) is

$$\frac{}{\langle v \parallel S\rangle \rightarrow \langle \theta_S(v) \parallel S\rangle} \tag{22.2.2}$$

where v is a variable. This says that evaluation of a variable is performed by getting the value of the variable out of the simple state. For example, relationship $\langle x \parallel \{x = 20\}\rangle \rightarrow \langle 20 \parallel \{x = 20\}\rangle$ follows from rule (22.2.2). Inference rule

$$\frac{}{\langle v \ := \ c; \ L \parallel S\rangle \rightarrow \langle L \parallel S[v := c]\rangle} \tag{22.2.3}$$

tells how to perform an assignment statement once the right-hand side has been simplified to a constant c. You simply add binding $v = c$ to the state and remove the assignment statement from the program. Rule (22.2.3) tells you, for example, that relationship $\langle$x := 2; y := 4; $\parallel \{z = 0\}\rangle \rightarrow \langle$y := 4; $\parallel \{x = 2, z = 0\}\rangle$ is true.

22.3. An example operational semantics

Figures 22.3 and 22.4 show an operational semantics for a very small programming language. Expressions include constants and variables, and allow the use of binary operators such as

$$
\begin{array}{ll}
\quad \langle x := 0;\ \textbf{while}\ x < 2\ \textbf{do}\ x := x + 1;\ \textbf{endwhile} & \| \{\} \rangle \\
\rightarrow \langle \textbf{while}\ x < 2\ \textbf{do}\ x := x + 1;\ \textbf{endwhile} & \| \{x = 0\} \rangle \\
\rightarrow \langle \textbf{if}\ x < 2\ \textbf{then}\ x := x + 1;\ \textbf{while}\ x < 2\ \textbf{do}\ x := x + 1;\ \textbf{endwhile endif} & \| \{x = 0\} \rangle \\
\rightarrow \langle \textbf{if}\ 0 < 2\ \textbf{then}\ x := x + 1;\ \textbf{while}\ x < 2\ \textbf{do}\ x := x + 1;\ \textbf{endwhile endif} & \| \{x = 0\} \rangle \\
\rightarrow \langle \textbf{if}\ \text{true}\ \textbf{then}\ x := x + 1;\ \textbf{while}\ x < 2\ \textbf{do}\ x := x + 1;\ \textbf{endwhile endif} & \| \{x = 0\} \rangle \\
\rightarrow \langle x := x + 1;\ \textbf{while}\ x < 2\ \textbf{do}\ x := x + 1;\ \textbf{endwhile} & \| \{x = 0\} \rangle \\
\rightarrow \langle x := 0 + 1;\ \textbf{while}\ x < 2\ \textbf{do}\ x := x + 1;\ \textbf{endwhile} & \| \{x = 0\} \rangle \\
\rightarrow \langle x := 1;\ \textbf{while}\ x < 2\ \textbf{do}\ x := x + 1;\ \textbf{endwhile} & \| \{x = 0\} \rangle \\
\rightarrow \langle \textbf{while}\ x < 2\ \textbf{do}\ x := x + 1;\ \textbf{endwhile} & \| \{x = 1\} \rangle \\
\rightarrow \langle \textbf{if}\ x < 2\ \textbf{then}\ x := x + 1;\ \textbf{while}\ x < 2\ \textbf{do}\ x := x + 1;\ \textbf{endwhile endif} & \| \{x = 1\} \rangle \\
\rightarrow \langle \textbf{if}\ 1 < 2\ \textbf{then}\ x := x + 1;\ \textbf{while}\ x < 2\ \textbf{do}\ x := x + 1;\ \textbf{endwhile endif} & \| \{x = 1\} \rangle \\
\rightarrow \langle \textbf{if}\ \text{true}\ \textbf{then}\ x := x + 1;\ \textbf{while}\ x < 2\ \textbf{do}\ x := x + 1;\ \textbf{endwhile endif} & \| \{x = 1\} \rangle \\
\rightarrow \langle x := x + 1;\ \textbf{while}\ x < 2\ \textbf{do}\ x := x + 1;\ \textbf{endwhile} & \| \{x = 1\} \rangle \\
\rightarrow \langle x := 1 + 1;\ \textbf{while}\ x < 2\ \textbf{do}\ x := x + 1;\ \textbf{endwhile} & \| \{x = 1\} \rangle \\
\rightarrow \langle x := 2;\ \textbf{while}\ x < 2\ \textbf{do}\ x := x + 1;\ \textbf{endwhile} & \| \{x = 1\} \rangle \\
\rightarrow \langle \textbf{while}\ x < 2\ \textbf{do}\ x := x + 1;\ \textbf{endwhile} & \| \{x = 2\} \rangle \\
\rightarrow \langle \textbf{if}\ x < 2\ \textbf{then}\ x := x + 1;\ \textbf{while}\ x < 2\ \textbf{do}\ x := x + 1;\ \textbf{endwhile endif} & \| \{x = 2\} \rangle \\
\rightarrow \langle \textbf{if}\ 2 < 2\ \textbf{then}\ x := x + 1;\ \textbf{while}\ x < 2\ \textbf{do}\ x := x + 1;\ \textbf{endwhile endif} & \| \{x = 2\} \rangle \\
\rightarrow \langle \textbf{if}\ \text{false}\ \textbf{then}\ x := x + 1;\ \textbf{while}\ x < 2\ \textbf{do}\ x := x + 1;\ \textbf{endwhile endif} & \| \{x = 2\} \rangle \\
\rightarrow \langle & \| \{x = 2\} \rangle
\end{array}
$$

Figure 22.2: **Evaluation sequence involving a loop.** This figure shows a longer example of an evaluation sequence for program fragment

$x := 0;$
while $x < 2$ **do**
  $x := x + 1;$
endwhile

starting in an empty simple state. As you can see, the semantics provides a great deal of detail about exactly what happens during computation. The iteration expressed by the while-loop is modeled by copying the body of the loop in front of the loop, and wrapping it all inside an if-statement, using the fact that statement **while** B **do** S **endwhile** is equivalent to

if B **then**
  S
  while B **do** S **endwhile**
endif

This evaluation sequence follows the general rules in Figures 22.3 and 22.4.

Rule Name	Rule
(var 1)	$\overline{\langle v \parallel S\rangle \rightarrow \langle \theta_S(v) \parallel S\rangle}$
(op 1)	$\overline{\langle c_1 \text{ op } c_2 \parallel S\rangle \rightarrow \langle \text{value}(c_1 \text{ op } c_2) \parallel S\rangle}$
(op 2)	$\dfrac{\langle \alpha \parallel S\rangle \rightarrow \langle \gamma \parallel T\rangle}{\langle \alpha \text{ op } \beta \parallel S\rangle \rightarrow \langle \gamma \text{ op } \beta \parallel T\rangle}$
(op 3)	$\dfrac{\langle \alpha \parallel S\rangle \rightarrow \langle \gamma \parallel T\rangle}{\langle c \text{ op } \alpha \parallel S\rangle \rightarrow \langle c \text{ op } \gamma \parallel T\rangle}$

Figure 22.3: **An operational semantics of expressions in a very small programming language.** Symbols c, c_1 and c_2 stand for arbitrary constants; v stands for a variable; α, β and γ stand for arbitrary expressions. Notation value(c_1 op c_2) indicates the value of an expression. For example, value$(3 + 5) = 8$, and value$(3 < 5) =$ true.

Rule (var 1) says to fetch the value of a variable from the state when you encounter a variable.

Taken together, rules (op 1), (op 2) and (op 3) say that, to evaluate expression (A op B), first evaluate A, then evaluate B, and finally perform the operation that is called for.

+ and <. For example, $x < 5$ is an expression, indicating a test that produces either **true** or **false** as its result. Any collection of constants and binary operators can be handled by this semantics, as long as the concept of the value of an expression found in rule (op 1) is already defined. In general, value(a op b) is assumed to tell the value of expression a op b, where a and b are constants. For example value$(4 + 11) = 15$. (A careful definition of value(a op b) for each operator op is part of the semantic definition, but we will not go into that here.)

Statements are (1) assignments of the form v **:=** e**;**, (2) conditionals of the form **if** e **then** S_1 **else** S_2 **endif** and **if** e **then** S_1 **endif**, and (3) loops of the form **while** e **do** S_1 **endwhile**. In the conditional and loop statements, each of S_1 and S_2 can be a sequence of statements.

You should be able to see the rules in Figures 22.3 and 22.4 at work in Figure 22.2. Figure 22.5 shows a proof of relationship $\langle y := x + 1; \parallel \{x = 8\}\rangle \rightarrow \langle y := 8 + 1; \parallel \{x = 8\}\rangle$ using those inference rules.

Notice that it can takes several *inference* steps to derive one fact of the form $s \rightarrow t$. The inference steps are not performed by the program, and are not part of the computation. It only takes one atomic step, in our semantics, to get from $\langle y := x + 1; \parallel \{x = 8\}\rangle$ to $\langle y := 8 + 1; \parallel \{x = 8\}\rangle$, even though it takes us several steps to demonstrate that such a step will be made.

22.4. Exercises

22.1. If you are interested in how much time a particular computation takes, which kind of semantics would you find most useful, an operational semantics, a denotational

Rule Name	Rule
(assign 1)	$\dfrac{\langle \alpha \parallel S\rangle \to \langle \beta \parallel T\rangle}{\langle v := \alpha;\ L \parallel S\rangle \to \langle v := \beta;\ L \parallel T\rangle}$
(assign 2)	$\overline{\langle v := c;\ L \parallel S\rangle \to \langle L \parallel S[v := c]\rangle}$
(if 1)	$\overline{\langle \textbf{if } \text{true } \textbf{then } A \textbf{ else } B \textbf{ endif } L \parallel S\rangle \to \langle A\, L \parallel S\rangle}$
(if 2)	$\overline{\langle \textbf{if } \text{false } \textbf{then } A \textbf{ else } B \textbf{ endif } L \parallel S\rangle \to \langle B\, L \parallel S\rangle}$
(if 3)	$\dfrac{\langle \alpha \parallel S\rangle \to \langle \beta \parallel T\rangle}{\langle \textbf{if } \alpha \textbf{ then } A \textbf{ else } B \textbf{ endif } L \parallel S\rangle \to \langle \textbf{if } \beta \textbf{ then } A \textbf{ else } B \textbf{ endif } L \parallel T\rangle}$
(if 4)	$\overline{\langle \textbf{if } \text{true } \textbf{then } A \textbf{ endif } L \parallel S\rangle \to \langle A\ L \parallel S\rangle}$
(if 5)	$\overline{\langle \textbf{if } \text{false } \textbf{then } A \textbf{ endif } L \parallel S\rangle \to \langle L \parallel S\rangle}$
(if 6)	$\dfrac{\langle \alpha \parallel S\rangle \to \langle \beta \parallel T\rangle}{\langle \textbf{if } \alpha \textbf{ then } A \textbf{ endif } L \parallel S\rangle \to \langle \textbf{if } \beta \textbf{ then } A \textbf{ endif } L \parallel T\rangle}$
(while 1)	$\overline{\langle \textbf{while } \alpha \textbf{ do } A \textbf{ endwhile } L \parallel S\rangle \to \langle \textbf{if } \alpha \textbf{ then } A \textbf{ while } \alpha \textbf{ do } A \textbf{ endwhile endif } L \parallel S\rangle}$

Figure 22.4: **An operational semantics of statements in a very small programming language.** Symbol c stands for an arbitrary constant; v stands for a variable; α and β stand for arbitrary expressions; and A, B and L stand for sequences of (zero or more) statements.

The rules for assignment say that, to perform $v := E$, first evaluate E (assign 1), yielding some number c, then add $v = c$ to the state, and remove the assignment from the program, since it has been done (assign 2). You should be able to understand what the remaining rules say to do.

1. $\langle x \parallel \{x = 8\}\rangle \to \langle 8 \parallel \{x = 8\}\rangle$ [by (var 1)].
2. $\langle x + 1 \parallel \{x = 8\}\rangle \to \langle 8 + 1 \parallel \{x = 8\}\rangle$ [by (op 2), using fact 1].
3. $\langle y := x + 1; \parallel \{x = 8\}\rangle \to \langle y := 8{+}1; \parallel \{x = 8\}\rangle$ [by (assign 1), using fact 2].

Figure 22.5: **Proof that $\langle y := x + 1; \parallel \{x = 8\}\rangle \to \langle y := 8 + 1; \parallel \{x = 8\}\rangle$.** Notice that it can take several steps of reasoning to derive what a single step of the program does. You start with simple relationships and build up to larger ones.

semantics or an axiomatic semantics?

22.2. What distinguishes an axiomatic semantics from other kinds of semantics?

22.3. What distinguishes an operational semantics from a denotational semantics?

22.4. Some implementations of Fortran allow a program to change the value of a constant. For example, you can change constant 2.0 to be 3.0, causing statement `X = 2.0` to set variable X to 3.0. (Constants are stored in the memory, and you are allowed to change the memory that holds a constant.) How would this affect the definition of the semantics of Fortran? Would you call Fortran a well-principled language?

22.5. Running out of memory is usually fatal to a program. Should running out of memory be considered a semantic black hole?

22.6. Using the semantic rules of Figures 22.3 and 22.4, show the entire sequence of full states in the evaluation of statement

```
if x > 0 then
  y := y + 1;
else
  z := z + 1;
endif
```

starting in simple state {x = 1, y = 5, z = 10}, until the program fragment is done.

22.7. Using the semantic rules of Figures 22.3 and 22.4, show the entire sequence of full states in the evaluation of statement

```
while x < 3 do
  y := y + y;
  x := x + 1;
endwhile
```

starting in simple state {x = 1, y = 5}, until the program fragment is done.

22.8. Which of the following relations are true according to the rules in Figures 22.3 and 22.4?

(a) $\langle y + 1 \parallel \{y = 8\}\rangle \rightarrow \langle 9 \parallel \{y = 8\}\rangle$

(b) $\langle y + 1 \parallel \{y = 8\}\rangle \rightarrow \langle 8 + 1 \parallel \{y = 8\}\rangle$

(c) $\langle x := 4;\ y := 10;\ \parallel \{\}\rangle \rightarrow \langle y := 10;\ \parallel \{x = 4\}\rangle$

(d) $\langle \textbf{if}\ \ x < y\ \ \textbf{then}\ \ w := 4;\ \ \textbf{else}\ \ w := 5;\ \textbf{endif} \parallel \{x = 1, y = 2\}\rangle \rightarrow \langle w := 4;\ \parallel \{x = 1, y = 2\}\rangle$

(e) $\langle \textbf{if}\ \ x < y\ \ \textbf{then}\ \ z := 5;\ \ \textbf{else}\ \ z := 10;\ \textbf{endif} \parallel \{x = 1, y = 2\}\rangle \rightarrow \langle \textbf{if}\ \ 1 < y\ \ \textbf{then}\ \ z := 5;\ \ \textbf{else}\ \ z := 10;\ \textbf{endif} \parallel \{x = 1, y = 2\}\rangle$

(f) $\langle x := 2 + 3;\ \parallel \{x = 7\}\rangle \rightarrow \langle x := 5 \parallel \{x = 7\}\rangle$

(g) $\langle x := x + 1;\ \parallel \{x = 7\}\rangle \rightarrow \langle\ \parallel \{x = 8\}\rangle$

(h) $\langle x := 2 + 3;\ \parallel \{x = 7\}\rangle \rightarrow \langle x := 2{+}3;\ \parallel \{x = 7\}\rangle$

22.9. Using the rules of Figures 22.3 and 22.4, prove each of the relationships in the previous exercise that is true.

22.10. Using the notation of Section 22.2.6, provide semantic rules for a conditional expression (if a then b else c), which produces the value of expression b when expression a produces true, and of c when a produces false.

22.11. Using the notation of Section 22.2.6, provide semantic rules for the C++ or Java expression $x{+}{+}$, which produces the current value of variable x, but has the side effect of adding 1 to x. So if x currently has value 3, then expression $x{+}{+}$ has value 3, but evaluation of $x{+}{+}$ causes x to take on value 4.

22.12. Using the notation of Section 22.2.6, provide semantic rules for the C++ or Java expression $+{+}x$, which adds one to x and produces the new value x. So if x currently has value 3, then expression $+{+}x$ has value 4, and evaluation of $+{+}x$ causes x to get value 4.

22.5. Bibliographic notes

Winskel [102] and Pagan [77] cover different kinds of semantics of programming languages.

The Algol 68 report [97] gives a formal semantics of Algol 68, and its effects are discussed by Lindsey [66]. Milner, Tofte and Harper [71] give a formal semantics of ML. Winskel [102] describes different ways of describing programming language semantics. Pierce [79] explains operational semantics. Arnold, Joy, Steele and Bracha [7] give a much less mathematical definition of Java.

Chapter 23

Lambda Calculus and Denotational Semantics

This chapter explores methods of carefully describing the semantics of expressions, including semantics for languages that support higher-order functions. The semantics is based on a mathematical foundation called λ-calculus, which is a small programming language in its own right, and in addition to providing a semantic basis, it serves as an example of a simple but powerful programming language.

23.1. Denotational semantics

Chapter 22 explains how to give an operational semantics of expressions, based on showing the sequence of steps that are involved in doing an evaluation. An alternative approach is a *denotational* semantics, which avoids explaining how an expression is evaluated, and instead jumps directly to the result. The result of evaluating an expression is said the be the value that the expression *denotes*.

Denotational semantics is a kind of *model theory*. A model theory starts with a *model* that is assumed to be understood. For example, the model might be the set of integers, and you must already understand what integers are in order for a model theory based on them to be effective. Each expression is given a meaning, its *denotation*, that is a member of the model. For example, expression $3 + 5$ might denote the number 8 in the model.

If e is an expression, let $D(e)$ indicate the denotation of e. For example, $D(8-2) = 6$. We need general rules for defining the D function. An obvious rule concerns numbers and other constants. The symbol, or numeral, 5 denotes the number 5. That is,

$$D(n) = n \quad \text{for any numeral } n.$$

It is important to distinguish between the *numeral* n (just some symbol that occurs in an expression) and the *number* n (a concept from mathematics, or from the model). For example, the numeral 341 is a sequence of three characters. Its denotation, the number 341, is a mathematical concept that you might explain to someone using a pile of 341 toothpicks.

The denotation of $a+b$ can be defined in terms of the addition operation in the model. But equation $D(a+b) = a+b$ cannot be right, since a and b are not numbers, but expressions. Where + on the left-hand side is a symbol in the language (part of an expression), the +

on the right-hand side of the equation is the addition operator of mathematics (part of the model), and it does not add expressions; it adds numbers. Before computing the denotation of $a + b$, we need to know the denotations of expressions a and b. That leads to definition

$$D(a + b) = D(a) + D(b).$$

Other definitions immediately suggest themselves.

$$\begin{aligned} D(a * b) &= D(a) \cdot D(b) \\ D(a - b) &= D(a) - D(b) \\ D(a/b) &= \frac{D(a)}{D(b)} \end{aligned}$$

A general principle of denotational semantics is that the denotation of an expression is defined in terms of the denotations of its parts.

Denotational semantics hides details about how expressions are evaluated. The only thing that you care about is the answer that is computed. For example, expressions $3 + 3 + 3 + 3$ and 12 both denote the number 12, even if the shorter one, 12, is quicker to compute. In general, denotational semantics suppresses issues of performance of programs, concentrating on the answers that are ultimately produced. Part of the simplicity and elegance of a denotational semantics is that it takes you directly to the answer, ignoring the details of the intermediate steps.

Handling variables

There is a close relationship between a denotational semantics and an interpreter written in a functional style. Examination of the simple interpreter of Figure 11.4, written in Scheme, shows that, ignoring differences in notation, the interpreter is almost identical to the denotational semantics of the preceding subsection. To evaluate $a + b$, the interpreter calls itself recursively on a and b, then it adds the results.

The semantics defined in the preceding subsection, like the Scheme interpreter eval-expr, handles expressions that only contain constants and operators. But what about expressions that contain variables (or names), such as x and y? Expression $x + y$ can only be given a denotation if the values of x and y are given. That is, it can only be given meaning in a particular state, or *environment*, that defines bindings of a collection of symbols. For example, environment $\{x = 5, y = 8\}$ indicates bindings for x and y, but not for any other symbols. If η (the Greek letter eta) is an environment, write $\eta(x)$ for the value that η gives to name x. For example, if η is $\{x = 5, y = 8\}$, then $\eta(y) = 8$. If environment η does not provide a binding of x, then $\eta(x)$ is undefined.

An interpreter that handles variables is generally written by adding another parameter, indicating the environment. Similarly, the concept of the denotation of an expression needs to be changed to take environments into account. If η is an environment and e is an expression, then let $D(e, \eta)$ be the denotation of expression e subject to the bindings in environment η. For example, $D(x + y, \{x = 5, y = 8\}) = 13$. Definitions follow fairly easily. Here are just a few.

$$\begin{aligned} D(x, \eta) &= \eta(x) && \text{for a symbol } x \\ D(n, \eta) &= n && \text{for a numeral } n \\ D(a + b, \eta) &= D(a, \eta) + D(b, \eta) \end{aligned}$$

Any expression that refers to an unbound symbol has no denotation. For example, $D(x, \{\})$ is undefined.

Suppose that, in a certain language, expression `let` x `=` v `in` e is supposed to evaluate expression v, bind identifier x to the value of v, and then evaluate expression e, yielding the value obtained for e. For example, expression `let y = 30 + 2 in y + y` has value 64. The semantics of let-expressions is easy to describe using denotational semantics. If η is an environment, let $\eta/\{\mathrm{x} = \mathrm{v}\}$ be the environment that has all bindings of η, but additionally binds identifier x to v. So if η is $\{x = 3, y = 7\}$ then $\eta/\{z = 9\}$ is environment $\{x = 3, y = 7, z = 9\}$. If η already has a binding for x, then $\eta/\{x = v\}$ uses binding v for x, replacing the prior binding. Using that notation, the semantics of let-expressions is defined as follows.

$$D(\text{let } x = a \text{ in } b,\ \eta) = D(b,\ \eta/\{x = D(a, \eta)\}).$$

23.2. Introduction to Lambda-calculus

In order to have a viable model theory, you must start with a model, so an obvious first step in defining a denotational semantics is selection of an appropriate model. A key criterion for this selection is that the model must contain a value for each thing in the programming language that you want to describe. For simple expressions involving integers, the set of integers provides an excellent model.

But programming languages are much richer than that. Not only are there many different kinds of data values, such as lists and other data structures, but a particular concern is that some of the things that must be dealt with are not typical data values at all, but are functions. How can those be modeled? For example, if you write a factorial function in a program, you would like to provide a denotation for the factorial function itself, so that you can say exactly what the function that you have written *is* in the model. Obviously, the model has to contain the factorial function (or something that represents the factorial function) as one of its members. To be suitable for a higher-order language, the model must have functions that take functions and yield functions.

There is another serious problem in choosing a model. The point of model theory is that the model must be understood ahead of time, and the language is then explained in terms of that model. What if each programming language uses a different model for its semantics? Then part of the semantics of the language must be a careful description of the model. That is, you need yet another semantics of the model. If every concept in the programming language has simply been added directly to the model, then the model offers no real clarification; its semantics will be just as difficult as the semantics of the programming language.

For denotational semantics truly to work well, some model must be chosen that is rich enough that it can be used as the model for all of the concepts in almost any programming language. That is a tall order. Fortunately, there is such a model, called λ-calculus (lambda calculus). It was, in fact, λ-calculus that lead to the programming language Lisp, of which Scheme is a dialect.

Lambda-calculus uses notation $\lambda x.e$ for the function that, when run on parameter x, produces value e. For example, the function that squares its argument is written $\lambda x.x \cdot x$. Function $\lambda x.x$ is the identity function. In $\lambda x.e$, x is called the formal parameter and e is called the body.

A function can be applied to an argument. To indicate function application (or

$$
\begin{array}{rcl}
\underline{((\lambda f.(\lambda x.f(x))))\,(\lambda y.y\cdot y)}\,(6) & \to & \underline{(\lambda x.(\lambda y.y\cdot y)x)\,(6)} \\
 & \to & \underline{(\lambda y.y\cdot y)\,(6)} \\
 & \to & \underline{6\cdot 6} \\
 & \to & 36 \\
(\lambda x.(\lambda y.y+(x+1)))\,(9)\,\underline{((\lambda z.z)(12))} & \to & \underline{(\lambda x.(\lambda y.y+(x+1)))\,(9)}\,(12) \\
 & \to & (\lambda y.y+(\underline{9+1}))\,(12) \\
 & \to & \underline{(\lambda y.y+(10))\,(12)} \\
 & \to & \underline{(12)+(10)} \\
 & \to & 22
\end{array}
$$

Figure 23.1: **Two evaluations by substitution.** At each step, the subexpression that is being replaced is underlined. The fundamental rule of computation in λ-calculus is that a term of the form $(\lambda x.E)(A)$ can be replaced by E', the result of replacing x in E by A. So $(\lambda x.x+x)(z) \to z+z$. We also use elementary rules of arithmetic to simplify some expressions.

simply *application*), write the function to the left of the argument. That is, juxtaposition indicates function application. For example $(\lambda x.x)\,5$ indicates application of function $\lambda x.x$ to argument 5. (The result is 5.) Functions applications are evaluated by substitution. For example, $(\lambda x.x\cdot x)\,5 = 5\cdot 5$ by substituting 5 for x in the body of the function. Substitution is called a *reduction* in λ-calculus terminology, and computation is done by performing a sequence of reductions until no more reductions are possible. When A reduces to B, we write $A \to B$. For example, $(\lambda x.x\cdot x)\,5 \to 5\cdot 5$.

You can use λ notation and function application in arbitrary ways to build up expressions of λ-calculus. For example, $\lambda x.(x\,x)$ is the rather peculiar function that takes a parameter x (a function) and returns the result of applying function x to itself. By convention, application has higher precedence than dot, so $\lambda x.x\,x$ is read as if parenthesized $\lambda x.(x\,x)$, not as $(\lambda x.x)x$. Also, by convention, application associates to the left, so $A\,B\,C$ is understood as $(A\,B)C$.

Functions are values, and a function can produce another function as its result. Function $\lambda f.(\lambda x.f(x))$, for example, takes parameter f (a function) and produces result $\lambda x.f(x)$ (also a function). Evaluating $(\lambda y.(\lambda r.r\,y))(5)$ by performing a substitution yields $\lambda r.r\,5$. Each of functions $\lambda f.(\lambda x.f(x))$ and $\lambda y.(\lambda r.r\,y)$ is *curried*, taking its parameters one at a time. Lambda notation encourages curried functions by defining λ to associate to the right. For example, the function $((\lambda f.(\lambda x.f(x))))$ can be written with fewer parentheses as $\lambda f.\lambda x.f\,x$. Figure 23.1 shows two sample evaluations by substitution, also employing basic rules of arithmetic.

Be careful about parenthesizing terms. Function application is *not* associative. That is $(a\,b)\,c$ is not, in general, the same as $a\,(b\,c)$. It would not be correct to begin an evaluation as follows.

$$((\lambda f.(\lambda x.f(x))))\,\underline{(\lambda y.y\cdot y)\,6} \to ((\lambda f.(\lambda x.f(x))))\,(6\cdot 6)$$

Substituting $6\cdot 6 = 36$ and continuing this faulty evaluation yields term $\lambda x.36\,x$, which makes no sense, since it is using 36 as a function.

23.3. A more careful treatment of Lambda-calculus

The preceding section gave an informal introduction to λ-calculus. This section gives a more careful development of the fundamental ideas.

Terms: the syntax of λ-calculus

Expressions in λ-calculus are called *terms*, and are built using the following rules.

1. A variable is a term. Any identifier can be used as a variable; we will use x, y and z as variable names.
2. If x is a variable and e is a term, then $\lambda x.e$ is a term. This kind of term is called an *abstraction*.
3. If a and b are terms then $a\,b$ is a term. This kind of term is called an *application*.

Parentheses can be used freely to show structure. Application (juxtaposition) has higher precedence than abstraction; application associates to the left, and abstraction associates to the right.

Only the forms (1)-(3) are supported by pure λ-calculus. Sometimes, though, it is useful to imagine that some other kinds of terms are also available. For example, terms that are numeric constants (3, 42, etc.) and elementary operations of arithmetic such as addition and subtraction are useful, especially for illustrating the ideas of λ-calculus. We will use those kinds of terms for discussions, but will show how to do without them later.

Free and bound variables

Each occurrence of a variable in a term is either *bound* or *free* or *formal*. A formal occurrence is one that occurs immediately after λ. For example, the x in $\lambda x.y$ is a formal variable. A bound occurrence of variable x is one that is not formal, but occurs within term e of larger term $\lambda x.e$. A bound variable refers to a function parameter, and will be substituted for when the function is applied. All other occurrences of variables are free. For example, in $\lambda x.y\,x$, variable y is free (since it does not occur inside any larger term of the form $\lambda y.e$) and the last x is bound (since it occurs with the scope of λx).

Lambda calculus allows *shadowing*. A bound variable x is said to be bound by the innermost abstraction that has that same variable as a formal parameter. So, in $\lambda x.(\lambda x.x)$, the bound occurrence of x is bound by the inner abstraction $\lambda x.x$, not by the outer, or first, one. The inner formal x shadows the outer one.

You can think of formal, bound and free variables in terms related to programming languages. A formal variable is like a formal parameter in a function definition. A bound variable is a formal parameter used inside a function definition. A free variable is like a global (or nonlocal) variable, not defined inside the function. There is no notion in λ-calculus that corresponds to a local variable that is neither a function parameter nor a global variable.

Reduction and computation

Computation involves substitutions. The substitution rule, called the rule of *β-reduction*, calls for replacing an an actual parameter for a formal parameter. Say that term $(\lambda x.e)a \rightarrow e'$ if e' is the result of replacing each free occurrence of x in e by (a). For example,

$(\lambda y.y\,x)\,z \rightarrow (z)x$. Notice that λy has been removed and y has been replaced by (z) in term $y\,x$, yielding $(z)x$.

You must put parentheses around the substituted value unless leaving them out does not change the structure of the term. For example, $(\lambda x.x\,x)(y\,y) \rightarrow (y\,y)(y\,y)$. It would not be correct to say that $(\lambda x.x\,x)(y\,y) \rightarrow y\,y\,y\,y$, since $y\,y\,y\,y$ implicitly means $((y\,y)y)y$, not $(y\,y)(y\,y)$. Remember that juxtaposition is not an associative operator in λ-calculus, and that, by convention, it is left-associative.

There is an important restriction, discussed below, on when β-reductions can be done.

Conversions and equivalence of terms

Computation only goes one way; if $A \rightarrow B$, you convert A into B, not B into A. There is a notion of *equivalence* of terms, though, that is more general, and that is symmetric ($x = y$ just when $y = x$). It is based on conversions among terms. If there is a conversion between two terms, the two terms are thought of as standing for the same thing, or as being equivalent, just as the terms 2+4 and 3+3 stand for the same value, even though they are different terms. There are three kinds of conversions.

1. The names that you choose for formal parameters of functions are arbitrary, and do not affect the meaning of a term. Accordingly, the first kind of conversion, called an *α-conversion*, consists of nothing but renaming a formal parameter. Write $s \equiv_\alpha t$ to indicate that s can be transformed into t by renaming a formal parameter of a function. For example, $\lambda x.x \equiv_\alpha \lambda y.y$.

   More precisely, say that $\lambda x.e \equiv_\alpha \lambda y.e'$ if e' is the result of replacing each free occurrence of x in e by y. (The occurrences of x that are free in term e by itself are just those that are bound by λx in $\lambda x.e$.) There is an important restriction, discussed below, on when an α-conversion can be done.

2. The second kind of conversion is called a *β-conversion*, which is a symmetric version of β-reduction. Say that $s \equiv_\beta t$ provided either $s \rightarrow t$ or $t \rightarrow s$.

3. The third kind of conversion is less obvious than the first two, and is called an *η-conversion*. Consider term $\lambda x.f(x)$. On input x, this function produces result $f(x)$. That, of course, is exactly what function f does; on input x, function f produces result $f(x)$. So $\lambda x.f(x)$ is just the same function as f. Say that $\lambda x.f(x) \equiv_\eta f$ and, by symmetry, $f \equiv_\eta \lambda x.f(x)$. You can choose any name for the formal parameter.

In general, conversions are not required to be done on entire terms, but can also be done on subterms. Suppose, for example, that $a \equiv_\beta b$, and that a is a subterm of a larger term A. If B is the term that results from term A by replacing a by b, then we say that $A \equiv_\beta B$. Similar rules apply to the other conversions. For example $\lambda x.\lambda y.x + y \equiv_\alpha \lambda x.\lambda v.x + v$ because the α conversion $\lambda y.x + y \equiv_\alpha \lambda v.x + v$ can be done on subterm $\lambda y.x + y$.

The conversions discussed so far perform only one step. A more general notion of conversion allows several steps to be performed. Say that $A \equiv B$, or A is *equivalent* to B, if there is a sequence of terms $A_0, A_1, \ldots, A_n$, for $n \geq 0$, where $A_0 = A$, $A_n = B$ and for $i = 1, \ldots, n$, either $A_{i-1} \equiv_\alpha A_i$ or $A_{i-1} \equiv_\beta A_i$ or $A_{i-1} \equiv_\eta A_i$. That is $A \equiv B$ if A can be converted to B by a sequence of zero or more basic conversions. Also, it is convenient to extend the notion of α-conversion to allow a sequence of conversions, so we write $A \equiv_\alpha B$ if A can be converted to B by doing zero or more α conversions.

Say that $A \rightarrow^* B$ (A reduces to B) if there is a sequence of terms $A_0, A_1, \ldots, A_n$ where $A_0 = A$, $A_n = B$ and for $i = 1, \ldots, n$, either $A_{i-1} \equiv_\alpha A_i$ or $A_{i-1} \rightarrow A_i$.

Restrictions on reductions and conversions

There is an important restriction on when α-conversions and β-reductions can be done. Consider term $\lambda x.\lambda y.x(y)$. If parameter x is renamed to y, you get $\lambda y.\lambda y.y(y)$. That is not the same thing, since now both bound occurrences of y are bound by the inner λ. This situation is called a *capture*. The inner function $\lambda y.x(y)$ captures x for itself when x is replaced by y.

Capture can also occur as the result of substitutions for formal parameters. Consider function $f = \lambda x.\lambda y.x + y$, a curried function that produces the sum of its two parameters. Imagine that you have, in your environment, a variable y whose value is 40. You would like to compute $y + 2$, so you write $(f\ y)\,2$, which should yield 42. But f is the function $\lambda x.\lambda y.x + y$, and substituting y for x in the body $\lambda y.x + y$ of f yields $f\ y = \lambda y.y + y$. Applying that function to 2 yields $2 + 2$, or 4. The dependence on the variable y in the environment has disappeared! The problem is that the substitution put free variable y into a context where it was captured by the λ, and became bound.

(Capture, or the lack of capture, really, shows up in programming languages, and it often results in confusion among beginning programmers. If two functions each have a local variable called x, a beginning programmer might presume that storing a value in one of those variables (say, inside function f) will cause another one of those variables (say, inside function g) to change. The beginning programmer is hoping that one function captures the other's variable, and is surprised when it is does not.)

You are only allowed to perform α-conversions and β-reductions when they do not result in any capture. A free variable must not become bound, and a bound variable must not become bound by a different λ. If a β-reduction would result in a capture, you must rename variables, by doing α-conversions, before performing the β-reduction. For example,

$$\begin{array}{rcl} (\lambda x.\lambda y.x + y)\ y\ 2 & \equiv_\alpha & (\lambda x.\lambda z.x + z)\ y\ 2 \\ & \rightarrow & (\lambda z.y + z)\ 2 \\ & \rightarrow & y + 2 \end{array}$$

23.4. Fundamental data types

Integers and elementary operations

Recall that the motivation for studying λ-calculus is to present a model in which a large variety of concepts are already present. It is not acceptable just to add each new concept to the model whenever a new concept is needed. Instead, new concepts should be modeled in terms of old concepts.

Until now, we have presumed that integer constants and arithmetic operations were available as primitive things. But, really, we should look for integers already present, in terms of what has already been defined. Although pure λ-calculus has no integers per se, it does have terms that can act as stand-ins for integers, and so can model integers. For simplicity, we will only explore nonnegative integers by defining, for each nonnegative integer n, a term $\bar{n}$ of λ-calculus that represents n. Here are the first few of what are called the

Church numerals.

$$\begin{aligned}
\bar{0} &= \lambda f.\lambda x.x \\
\bar{1} &= \lambda f.\lambda x.f\,x \\
\bar{2} &= \lambda f.\lambda x.f(f\,x) \\
\bar{3} &= \lambda f.\lambda x.f(f(f\,x)) \\
\bar{4} &= \lambda f.\lambda x.f(f(f(f\,x)))
\end{aligned}$$

The pattern continues. Using $f^n(x)$ to stand for $f(f(...(f(x))))$ (with n occurrences of f), the term $\bar{n}$ is $\lambda f.\lambda x.f^n(x)$. Notice that $\bar{n}(f)(x) = f^n(x)$.

To say that the Church numerals are reasonable stand-ins for integers, we need to be able to define basic operations on integers, making them work in the model. One of the most basic is the successor function, which adds one to an integer. It can be defined by

$$\text{succ} = \lambda m.\lambda f.\lambda x.f(m\,f\,x). \tag{23.4.1}$$

To demonstrate that the definition of succ is correct, it suffices to show that $\text{succ}(\bar{n}) = \overline{n+1}$. Subterms that are about to be reduced are underlined.

$$\begin{aligned}
\text{succ}(\bar{n}) &= \underline{(\lambda m.\lambda f.\lambda x.f(m\,f\,x))(\bar{n})} \\
&\to \lambda f.\lambda x.f(\bar{n}f\,x) \\
&= \lambda f.\lambda x.f(\underline{(\lambda u.\lambda v.u^n(v))\,f}\,x) \\
&\to \lambda f.\lambda x.f(\underline{(\lambda v.f^n(v))x}) \\
&\to \lambda f.\lambda x.f(f^n(x)) \\
&= \lambda f.\lambda x.f^{n+1}(x)) \\
&\equiv_\alpha \overline{n+1}
\end{aligned}$$

To define the addition function, *plus*, remember that $\bar{n}f$ is the same as f^n; it calls for applying function f a total of n times. Since $f^{n+m}(x) = f^n(f^m(x)) = (\bar{n}f)(\bar{m}fx)$, a reasonable definition of plus is

$$\text{plus} = \lambda n.\lambda m.\lambda f.\lambda v.(n\,f)(m\,f\,v).$$

To show that this works, try it with a general $\bar{n}$ and $\bar{m}$.

$$\begin{aligned}
\text{plus}\ \bar{n}\ \bar{m} &= \underline{(\lambda n.\lambda m.\lambda f.\lambda v.(n\,f)(m\,f\,v))\ \bar{n}}\ \bar{m} \\
&\to \underline{(\lambda m.\lambda f.\lambda v.(\bar{n}\,f)(m\,f\,v))\,\bar{m}} \\
&\to \lambda f.\lambda v.(\bar{n}\,f)(\bar{m}\,f\,v) \\
&= \lambda f.\lambda v.(\bar{n}\,f)(\underline{(\lambda x.\lambda y.x^m(y))\,f}\,v) \\
&\to \lambda f.\lambda v.(\bar{n}\,f)(\underline{(\lambda y.f^m(y))\,v}) \\
&\to \lambda f.\lambda v.(\bar{n}\,f)(f^m(v)) \\
&= \lambda f.\lambda v.(\underline{(\lambda x.\lambda y.x^n(y))\,f})(f^m(v)) \\
&\to \lambda f.\lambda v.\underline{(\lambda y.f^n(y))(f^m(v))} \\
&\to \lambda f.\lambda v.f^n(f^m(v)) \\
&= \lambda f.\lambda v.f^{n+m}(v) \\
&\equiv_\alpha \overline{n+m}
\end{aligned}$$

Integer multiplication is implemented by function *product*, defined as follows.

$$\text{product} = \lambda m.\lambda n.\lambda f.m(nf) \tag{23.4.2}$$

That is, perform a nested loop; repetition n times is done m times.

Predecessor and subtraction

Subtraction is tricky to do. The starting point is a function *pred* (for predecessor) that subtracts one from a number. Since we are only handling nonnegative numbers, define $\text{pred}(\bar{0}) = \bar{0}$. The intuition is to write a loop whose body is performed n times, but that produces $f^{n-1}(x)$. Such a loop might go something like this.

```
A = x;
B = x;
for i = 1,..., n do
  B = A;
  A = f(A);
end for
```

At the end of the loop, B contains $f^{n-1}(x)$ (or just x if $n = 0$). The two variables A and B can be encoded as an ordered pair, and the body of the loop can be carried out by a function that takes one ordered pair and produces the next one. That requires a function *pair* for creating ordered pairs (where (pair x y) represents the ordered pair (x, y)), and functions *left* and *right* for getting the parts of a pair (where left(pair x y) $\equiv x$ and right(pair x y) $\equiv y$). Those functions are defined later in this section.

Suppose that $g(x, y) = (f(x), x)$, where g deliberately ignores y. The parameter of g is an ordered pair, as encoded by function *pair*. Then

$$\begin{aligned} g(x, y) &= (f(x), x) \\ g(g(x, y)) &= (f(f(x)), f(x)) \end{aligned}$$

and, in general, $g^n(x, y) = (f^n(x),\ f^{n-1}(x))$, so $f^{n-1}(x) = \text{right}(g^n(x, y)) = \text{right}(\bar{n}\, g\, (x, y))$. The following definition of pred uses that idea. Since the value of $g^n(x, y)$ is independent of y, it is just as well to compute $g^n(x, x)$, and that is what is done. Function g is not written directly, but is produced as the result of another function; g is predhelp(f).

$$\begin{aligned} \text{predhelp} &= \lambda f.\lambda p.\text{pair } (f(\text{left } p)) \text{ (left } p) \\ \text{pred} &= \lambda n.\lambda f.\lambda x.\text{right}(n \text{ (predhelp } f) \text{ (pair } x\ x)) \end{aligned}$$

Subtraction can be defined by repeatedly taking the predecessor. It is left as an exercise.

Structured data

An ordered pair can be constructed by a function that takes three parameters. The first two parameters are the two members of the pair. The third parameter is a selector, which tells which member of the pair is desired. The pair constructor, *pair*, is defined as follows.

$$\text{pair} = \lambda x.\lambda y.\lambda s.s\, x\, y \tag{23.4.3}$$

Term (pair 5 7) models the ordered pair (5,7). Evaluation yields

$$\begin{aligned} \text{pair } 5\ 7 &= (\lambda x.\lambda y.\lambda s.s\ x\ y)\ 5\ 7 \\ &\to (\lambda y.\lambda s.s\ 5\ y)\ 7 \\ &\to (\lambda s.s\ 5\ 7) \end{aligned}$$

The pair is represented by a function that is waiting for a selector s. Notice that the selector is applied to 5 and 7, in that order, in the body s 5 7. Here are two functions, *left* and *right*, that take the left and right-hand member of an ordered pair, respectively, by supplying the pair with a suitable selector s.

$$\begin{aligned} \text{left} &= \lambda p.p(\lambda x.\lambda y.x) \\ \text{right} &= \lambda p.p(\lambda x.\lambda y.y) \end{aligned}$$

For example,

$$\begin{aligned} \text{left(pair } 5\ 7) &\to^* (\lambda p.p(\lambda x.\lambda y.x))(\lambda s.s\ 5\ 7) \\ &\to (\lambda s.s\ 5\ 7)(\lambda x.\lambda y.x) \\ &\to (\lambda x.\lambda y.x)\ 5\ 7 \\ &\to (\lambda y.5)\ 7 \\ &\to 5 \end{aligned}$$

Modeling lists is left as an exercise.

23.5. Control structures

We have seen that λ-calculus can model basic data values. But can it provide control structures, such as if-then-else and recursion? It turns out that both of those can be modeled without adding anything new. Conditionals are fairly simple to model. Recursion is more involved, and is covered in the next section.

The conditional function *cond* is defined so that (cond $x\,y\,z$) is y when x is true, and is z when x is false. The conditional function and associated boolean values *true* and *false* can be defined as follows.

$$\begin{aligned} \text{cond} &= \lambda x.\lambda y.\lambda z.x\,y\,z \\ \text{true} &= \lambda x.\lambda y.x \\ \text{false} &= \lambda x.\lambda y.y \end{aligned}$$

(Compare with pair, left and right.) For example,

$$\begin{aligned} \text{cond true } M\ N &= (\lambda x.\lambda y.\lambda z.x\ y\ z)\ \text{true}\ M\ N \\ &\to (\lambda y.\lambda z.\text{true}\ y\ z)\ M\ N \\ &\to (\lambda z.\text{true}\ M\ z)\ N \\ &\to \text{true}\ M\ N \\ &= (\lambda x.\lambda y.x)\ M\ N \\ &\to (\lambda y.M)\ N \\ &\to M \end{aligned}$$

23.6. Recursion

Lambda-calculus provides no direct way to make recursive definitions. You are always free to give names to terms, as we have done for values such as false, true and cond, but you cannot use a name before it is defined.

Recursion amounts to solving equations

For ease of understanding, suppose that integers and a few basic operations on integers are already implemented. Write $m * n$ for the product of m and n and $m - n$ for their difference. Also assume that a predicate that tests whether a given number is zero has been implemented. For clarity, that test is written $(n = 0)$ (Expressing those functions is left as an exercise.) Now suppose that you want to define the factorial function as

$$\text{factorial}(n) = \text{cond } (n = 0)\ 1\ (n * \text{factorial}(n - 1))$$

That would be fine, except that it is not allowed by λ-calculus, since the definition uses factorial before it is defined. Another way to express the definition of factorial is

$$\text{factorial} = \lambda n.\text{cond } (n = 0)\ 1\ (n * \text{factorial}(n - 1)) \tag{23.6.1}$$

but that is still not allowed. But generalize Equation (23.6.1) to the following equation involving a variable X that stands for an unknown function.

$$X = \lambda n.\text{cond } (n = 0)\ 1\ (n * X(n - 1)) \tag{23.6.2}$$

Clearly, $X = \text{factorial}$ is a solution of Equation (23.6.2). (Substitute $X = \text{factorial}$ and you get equation (23.6.1).) So the problem of defining the factorial function can be expressed as finding a function X that satisfies Equation (23.6.2), and, in general, recursive definitions can be viewed as finding solutions to certain equations, where the solutions are functions. All we need is a way to solve those equations.

Fixed-point iterations

Computers are frequently used to solve equations. One way that is commonly employed is to express the equation in the form $x = f(x)$. That is, find a value x, called a *fixed point* of f, that f maps to itself. An example is computation of square roots. If $f(x) = (x + n/x)/2$, then it is easy to check that $\sqrt{n}$ is a fixed point of f. That is, $f(\sqrt{n}) = \sqrt{n}$. You can get better and better approximations to $\sqrt{n}$ by defining $x_0 = n$ and $x_i = f(x_{i-1})$ for $i > 0$. So $x_1 = f(x_0)$, $x_2 = f(f(x_0))$, etc. Suppose, for example, that $n = 9$. The first few values x_0, x_1, ... are

$$\begin{aligned} x_0 &= 9 \\ x_1 &= 5 \\ x_2 &= 3.4 \\ x_3 &\approx 3.0235 \end{aligned}$$

The correct answer, of course, is the square root of 9, or 3. As more and more values in the sequence are computed, the answer gets closer and closer to 3. The starting value x_0 is

not terribly important, except that starting values closer to the answer cause the solution to converge to a correct answer faster.[1]

The same general approach can be used to find fixed points where the fixed point is not a number but a function. To compute the factorial function, define the function

$$H = \lambda f.\lambda n.\text{cond } (n = 0)\ 1\ (n * f(n-1)) \tag{23.6.3}$$

Equation $X = H(X)$ is the same as equation 23.6.2. So the factorial function is a fixed point of H. Suppose that a function f_0 is chosen as a first approximation of factorial. Exactly what f_0 does is irrelevant, and will have no bearing on the final answer. Compute $f_1 = H(f_0)$, $f_2 = H(f_1) = H(H(f_0))$, $f_3 = H(f_2) = H(H(H(f_0)))$, etc. For example,

$$f_1 = \lambda n.\text{cond } (n = 0)\ 1\ (n * f_0(n-1))$$

and $f_1(0) = 1$, regardless of what f_0 does. Since

$$f_2 = \lambda n.\text{cond } (n = 0)\ 1\ (n * f_1(n-1))$$

you can see that $f_2(1) = 1 * f_1(0) = 1$. Continuing shows that $f_3(2) = 2$ and $f_4(3) = 6$. A table of $f_4(n)$ for natural numbers n shows that $f_4(0) = 1$, $f_4(1) = 1$, $f_4(2) = 2$, $f_4(3) = 6$, and $f_4(n)$ depends on what f_0 does, for $n > 3$. As more and more copies of H are applied, the resulting function computes more and more of the factorial function. Function $f_6 = H(H(H(H(H(H(f_0))))))$ gets the correct values on all parameters up to 5, regardless of what f_0 is. To get the full factorial function would require performing H infinitely many times. But that is not really necessary. If you want to compute factorial(5), it suffices to compute function f_6 and to apply that function to 5. The answer 120 comes out in a finite amount of time. In fact, to compute factorial(5), you can use function f_k for any $k > 5$.

Recursion is handled in λ-calculus by creating a function, fix, that finds a fixed point of a function H by repeatedly applying H. For any function F, fix(F) is a fixed point of F. So $F(\text{fix}(F)) = \text{fix}(F)$. To get the factorial function, you define H by equation 23.6.3 and define factorial to be fix(H). The fix function can be defined as follows.

$$\text{fix} = \lambda f.(\lambda x.f(x\,x))(\lambda x.f(x\,x)) \tag{23.6.4}$$

Notice that, for any function F,

$$\text{fix}\,F \to (\lambda x.F(x\,x))(\lambda x.F(x\,x)) \tag{23.6.5}$$

Performing another β-reduction yields

$$\text{fix}\,F \to^* F((\lambda x.F(x\,x))(\lambda x.F(x\,x)))$$

Now, using (23.6.5) backwards yields

$$\text{fix}\,F \equiv F(\text{fix}\,F).$$

So (fix F) is a fixed point of F. The fix function just does repeated evaluation. Using the fixed point property, $\text{fix}\,F \equiv F(\text{fix}\,F) \equiv F(F(\text{fix}\,F)) \equiv F(F(F(\text{fix}\,F)))\ldots$ If you think of the inner (fix F) as the function f_0, you can see how the fixed point is found.

[1] Once you get close, this particular fixed-point iteration converges very fast, giving you about twice as many digits of precision at each step as you had at the previous step.

Recursion is a powerful tool, and, using fix, you can get the effect of recursive definitions in λ-calculus, even though recursion is not supported directly. It is generally believed that λ-calculus is powerful enough to express any function that can be expressed in any (realistically implementable) programming language. That observation is attributed to Alonzo Church, the creator of λ-calculus.

> **Church's Thesis.** All algorithmically computable functions are expressible in λ-calculus.

So λ-calculus models the kinds of things that are needed for programming languages, and is capable of expressing any function that can be defined in any language.

23.7. Evaluation order

How do you know when a computation in λ-calculus is finished? Say that a term of λ-calculus is in *normal form* if there are no β-reductions that can be applied to it. (If there is a β-reduction that is blocked because it would cause a capture, then the term is not in normal form. Appropriate renaming must be done, and the β-reduction carried out.) The goal of computation is to convert a term to normal form, called *normalizing* the term.

A term can be evaluated in various orders. Inside-out, or eager, evaluation is commonly employed. But outside-in evaluation, and lazy evaluation, can also be used. Evaluation order is partly a matter of efficiency; some orders can lead to a normal form in fewer steps than other orders. But your choice of evaluation order can also determine whether you will *ever* reach a normal form. The problem is that some terms have no normal form. Consider term $L = (\lambda x.x\,x)(\lambda x.x\,x)$. Performing a β-reduction shows that $L \to L$. Any attempt to evaluate L by performing β-reductions will never terminate. Now suppose that you want to evaluate term (cond true $\bar{0}$ L). If you use the rule for cond first, you get answer $\bar{0}$, which is in normal form. But if instead you decide to evaluate L first, you never reach that answer, and get stuck in an infinite computation.

Does there exist an evaluation strategy that finds a normal form whenever one exists? There is, and a simple one is left-to-right, outside-in evaluation. Define a *redex* (short for reducible expression) to be any term of the form $(\lambda x.e)(a)$, which can be replaced by performing a β-reduction. Suppose that a term contains more than one redex. An *outermost* redex is one that is not contained in any other redexes as a subterm. *Normal order* evaluation always selects the leftmost outermost redex, and performs a β-reduction on that subterm. Normal order evaluation reaches a normal form whenever it is possible to reach a normal form. (Proof of this fact is left to more in-depth studies of λ-calculus.)

The remainder of this chapter gives an equational definition of an evaluator that converts a term to normal form using normal order evaluation. The evaluator forms the basis of an operational semantics for λ-calculus, since it indicates how to perform computations, step by step. But the equational nature of the definition means that the steps are implicit, not explicitly stated as they would be using rules of inference. For simplicity, some of the details of the evaluator, such as dealing with capture, are only stated, not written out using equations.

Head-normal form

Head-normal form is a form that is not quite normal form, but is a good start toward it. Say that a term is in head-normal form if it has the form $\lambda x_1.\lambda x_2.\ \ldots \lambda x_m.y t_1 \ldots t_n$ for $m \geq 0$

$$
\begin{array}{rcll}
\text{hnf}(\lambda x.t) & = & \lambda x.\text{hnf}(t) & \text{(abstraction)} \\
\text{hnf}(t_1\ t_2) & = & \text{hnfhelp}(\text{hnf}(t_1)\ t_2) & \text{(application)} \\
\text{hnf}(x) & = & x & \text{(variable)} \\
\text{hnfhelp}((\lambda x.t)z) & = & \text{hnf}(t/\{x \leftarrow z\}) & \\
\text{hnfhelp}(r\ s) & = & r\ s & (r \text{ not an abstraction})
\end{array}
$$

Figure 23.2: **Definition of function** hnf**, which converts a term to head-normal form.** Notation $t/\{x \leftarrow z\}$ indicates the result of substituting z for all free occurrences of x in t. The first equation for hnfhelp presumes that no capture results from this substitution. If a capture would result, you must rename variables before performing the substitution.

$$
\begin{array}{rcll}
\text{nf}(t) & = & \text{nfhelp}(\text{hnf}(t)) & \\
\text{nfhelp}(\lambda x.t) & = & \lambda x.\text{nfhelp}(t) & \text{(abstraction)} \\
\text{nfhelp}(t_1\ t_2) & = & (\text{nfhelp}(t_1))\ (\text{nf}(t_2)) & \text{(application)} \\
\text{nfhelp}(x) & = & x & \text{(variable)}
\end{array}
$$

Figure 23.3: **Definition of function** nf**, which converts a term to normal form.**

and $n \geq 0$, where y is a variable and $t_1, \ldots, t_n$ are arbitrary terms. That is, it consists of zero or more λ's, then a variable name (which can be one of $x_1, \ldots, x_m$ or can be something else) then a sequence of zero or more terms $t_1, \ldots, t_n$. For example, $\lambda x.x((\lambda y.y)z)$ is in head-normal form, with $m = 1$ and $n = 1$. Term t_1 is $((\lambda y.y)z)$. You are not required to have any λ's. For example, term xy is in head-normal form. (It is also in normal form.)

The normal form of $\lambda x_1.\lambda x_2.\ldots\lambda x_m.y\,t_1\ldots t_n$, if it exists, is clearly $\lambda x_1.\lambda x_2.\ldots\lambda x_m.y\,v_1\ldots v_n$, where v_i is the normal form of t_i, for $i = 1, \ldots, n$. No reductions can disturb the outer structure; they must all happen within the terms $t_1, \ldots, t_n$. (Application associates to the left. After evaluating terms $t_1, \ldots, t_n$, we have $\lambda x_1.\lambda x_2.\ldots\lambda x_m.(((y\,v_1)\,v_2)\ldots v_n)$. There are no β-reductions to do.) So to normalize a term u it suffices to convert u to head-normal form $\lambda x_1.\lambda x_2.\ldots\lambda x_m.yt_1\ldots t_n$, and then to normalize each of $t_1, \ldots, t_n$.

Figure 23.2 gives an equational definition of function hnf, which converts a term to head-normal form. To understand how it works, imagine a term that is not in head-normal form. If you eliminate all redundant parentheses, then you must have a term of the form $\lambda x_1.\lambda x_2.\ldots\lambda x_m.r\,t_1\ldots t_n$, for $m \geq 0$ and $n \geq 0$, where term r is an abstraction $(\lambda y.a)$. (If r is an application, then it is not the first thing in the list after removing redundant parentheses. If r is a variable, then this term is in head-normal form.) So a term that is not in head-normal form must have the form $\lambda x_1.\lambda x_2.\ldots\lambda x_m.(\lambda y.a)t_1\ldots t_n$, where $n > 0$. Conversion to head-normal form is just a matter of performing a β-reduction of $(\lambda y.a)t_1$ and recursively converting the result to head-normal form. That is what the definition of hnf does.

Normal order evaluation

Conversion to normal form is completed by converting each of the terms $t_1, \ldots, t_n$ to normal form. Figure 23.3 shows a definition of function nf, which converts a term to normal form (or runs forever trying).

Confluence

Reduction using normal order is guaranteed to find a normal form if one exists, but it does not necessarily give the most efficient route to the normal form. It uses outside-in evaluation, which often requires evaluating a term more than once. Suppose that you decide to use a different strategy, such as inside-out evaluation (which evaluates an application $(\lambda x.A)\,B$ by giving preference to redexes in B before substituting it into A), and then decide that doing so was not such a good idea. To find a normal form, will you have to undo what you have done, and start back at the beginning with normal order? It turns out that you never need to back up. Lambda-calculus has a property called *confluence*.

> **Confluence property.** Suppose that term t has a normal form n. All evaluation strategies that yield a normal form yield the same normal form n. Moreover, if t reduces to t' by a sequence of α and β reductions, then there is a way to reduce t' to n by a sequence of α and β reductions.

The confluence property shows that you can never get into a blind alley and need to back up. You can always switch to head-normal evaluation at a later time if you need to.

23.8. Exercises

23.1. Write a definition of the denotation of expression `if` A `then` B `else` C, which produces B when A is true and produces C when A is false. Use $D(E, \eta)$ notation.

23.2. An alternative notation for the denotation of an expression uses a relation, $\Rightarrow$, where $e \Rightarrow v$ indicates that the denotation of expression e is v. An extension includes an environment: write $\eta \vdash e \Rightarrow v$ to indicate that, when variables have the values indicated in environment η, the denotation of expression e is v. For example, $\{x = 5\} \vdash x + 1 \Rightarrow 6$.

Facts about $\Rightarrow$ can be expressed using rules of inference. For example, a rule for addition is

$$\frac{\eta \vdash a \Rightarrow u,\ \eta \vdash b \Rightarrow v}{\eta \vdash a + b \Rightarrow u + v}$$

Using this notation, write rules of inference for the denotation of n, $a * b$, x and (let $x = a$ in b), where x is a variable (bound by the environment), a and b are expressions, and n is a numeral.

23.3. Add parentheses to each of the following terms to show their structure. Add enough parentheses so that no rules of precedence or associativity are required to understand them.

(a) $\lambda x.x\,x\,y$

(b) $\lambda x.\lambda y.y\,x$

(c) $\lambda x.x\,\lambda y.y$

23.4. By carefully performing β-reductions, show that $\mathrm{succ}(\bar{2}) \rightarrow^* \bar{3}$, using function succ defined in Equation (23.4.1).

23.5. Show that $(\mathrm{product}\ \bar{2}\ \bar{3}) \rightarrow^* \bar{6}$, using Equation (23.4.2) as the definition of product.

23.6. Reduce each of the following to normal form. Equation (23.4.3) defines pair.

(a) $\bar{0}\,\bar{0}$.

(b) $\bar{1}\,\bar{0}$.

(c) pair $\bar{0}$ $\bar{2}$.

23.7. Let fix be the fixed-point operator defined in equation 23.6.4.

(a) Evaluate $\text{fix}\,g$ using normal order evaluation. (Here, g is a free variable, and has no value, so the result must still have g in it.)

(b) Show that if $\text{fix}\,g$ is evaluated using inside-out evaluation instead of normal order evaluation, the evaluation will not terminate.

(c) An alternative fixed-point operator fix' is defined by $\text{fix}' = \lambda f.(\lambda x.f(\lambda y.x\,x\,y))(\lambda x.f(\lambda y.x\,x\,y))$. Show that evaluation of $\text{fix}'\,g$ by inside-out evaluation terminates. (Operator fix' has added machinery to force evaluation to be delayed until it is needed.)

23.8. Show that function fix' from the preceding exercise is a fixed point operator. That is, $f(\text{fix}'f) = \text{fix}'f$.

23.9. Give a definition of a function z in λ-calculus so that $z(\bar{0}) = \text{true}$ and $z(\bar{n}) = \text{false}$ for all integers $n > 0$. (This is the function that tests whether a number is 0.) (Hint: You might find it useful to write a function that always produces false. Notice that if F is defined so that $F(x) = \text{false}$ for every x, then $F^n(x) = \text{false}$ for every $n > 0$. Try computing $\bar{n}Fv$. What is the answer when $n = 0$? What is it when $n > 0$?)

23.10. Define the subtraction function diff in λ-calculus so that $\text{diff}\ \bar{n}\ \bar{m} = \overline{n-m}$ when $n \geq m$, and $\text{diff}\ \bar{n}\ \bar{m} = \bar{0}$ when $n < m$. (Hint: It suffices to take n predecessors of m. Keep in mind that $\bar{n}$ is designed to repeat an operation n times, so there is a built-in loop mechanism.)

23.11. Give a definition of the exponentiation function $\text{pow}\ \bar{n}\ \bar{m} = \overline{n^m}$ for nonnegative integers in λ-calculus. (There is a very short definition. See if you can find it.)

23.12. Define an equality predicate that works on numbers. So $\text{equal}\ \bar{n}\ \bar{n}$ should produce true, and $\text{equal}\ \bar{n}\ \bar{m}$ should produce false when $n \neq m$.

23.13. Show how to model lists in λ-calculus. Model the empty list, and functions null, ::, head and tail. (null(x) = true if x is the empty list, and false otherwise.) Use the implementation of pairs as a guide. Since there are two kinds of lists (empty and nonempty), you might find a tag useful. A boolean value makes a convenient tag. How can you attach a tag to a value?

23.14. You can define the fix function in Cinnameg as follows.

```
{fix f x = f (fix f) x}
```

Define function H whose fixed point is the factorial function, and use fix to get its fixed point. (But call it h, since Cinnameg prefers lower case names.) Try it to see whether it computes factorials.

23.15. Suppose the preceding problem uses the following definition.

```
{fix f = f(fix f)}
```

That does not work. Why not? What effect does the extra parameter x have? Is there an easy way to modify the new definition to make it work, without introducing the extra parameter x?

23.16. The capture problem makes implementation of an interpreter for λ-calculus awkward. One approach to reducing the difficulty is to use a representation that does not give names to formal parameters at all. Instead, each bound variable is replaced by a nonnegative integer, indicating how many λ's need to be skipped over, moving outwards from the location of the number, to find the λ that binds this variable. For example, term $\lambda x.\lambda y.xy(\lambda z.xz)$ is represented as $\lambda.\lambda.1\,0(\lambda.2\,0)$. Use this representation of λ-terms to write a normal-order evaluator for λ-calculus.

Be careful, and try some examples by hand first. When you perform a substitution, some of the variables move into different contexts, and need their numbers changed. (Remember that a β-reduction can be performed on a subterm of a larger term, and the rest of the term provides the context.) You will find the following function useful. It modifies variable numbers in a term by adding n to each free variable number that is larger than m. k is a term that is variable number k.

$$\begin{aligned}
\text{renum}(m, n, k) &= k \text{ when } k < m \\
\text{renum}(m, n, k) &= k + n \text{ when } k >= m \\
\text{renum}(m, n, \lambda.t) &= \lambda.\text{renum}(m + 1, n, t) \\
\text{renum}(m, n, s\,t) &= \text{renum}(m, n, s)\ \text{renum}(m, n, t)
\end{aligned}$$

23.17. Give rules of inference that describe an operational semantics showing how to evaluate a term of λ-calculus. The rules should define a single-step relation $A \to B$ that involves performing one β-reduction. Write the semantics in two different ways.

(a) Use call-by-value, where a parameter is converted to normal form before passing it to a function. For example, to evaluate $(\lambda x.A)(B)$, first evaluate B. To evaluate $A\,B$, evaluate A first, up to the point where it is an abstraction.

(b) Use normal order evaluation, where outermost reductions are done first. That is, to evaluate $(\lambda x.A)(B)$, perform the reduction (replacing free occurrences of x in A by B) before evaluating B. To evaluate $A\,B$, evaluate A first until it is an abstraction.

23.9. Bibliographic notes

Gordon [41] Mitchell [72], Pagan [77], Scott and Strachy [87], Schmidt [85], Stoy [93] and Winskel [102] all describe denotational semantics. Lambda calculus was introduced by Church [26], and is covered by Barendregt [11] and Hindley and Seldin [51]. There is also a typed version of λ-calculus, as described by Barendregt [12]. Rosser [83] gives a history of λ-calculus. The idea for representing terms described in Exercise 23.16 is due to DeBruijn [35].

Chapter 24

Reasoning about Programs

24.1. Reasoning about correctness of programs

How can you convince yourself that an imperative program is correct? How do you know that it does what you expect it to do? An obvious approach is to test it. Clearly, testing is an important part of creating a program, but it cannot be the entire basis for justifying the correctness of a program, for the following reasons.

1. Testing can show the presence of mistakes, but it cannot guarantee their absence. Testing inevitably misses some cases where errors might occur. So testing does not really convince you that your program is free of errors.

2. If testing is the only way of understanding whether or why a program works, then how was the program written? Was it written entirely by trial and error? That is very unlikely. You must have had in mind some reason for writing the program the way you did, and you must have used some form of correctness reasoning that had nothing to do with testing. It might help if that reasoning could be understood in more than strictly intuitive terms.

Another approach to convincing yourself that a program is correct is to perform a hand simulation. That goes directly to the underlying operational semantics. To see that a program is correct, you follow the sequence of states through which the program takes a computer. As a start, you can perform a hand simulation on specific test inputs. To test a factorial function, for example, you might hand simulate it to compute the factorial of 3. Unfortunately, just trying a few cases by hand cannot, in itself, convince you that the program works for all inputs, any more than doing those tests on a computer can convince you.

So, after trying some test cases, you might try doing a hand simulation on *generic* data, where you do not select specific values for the inputs, but leave them unknown. (If the input is an integer, n, just call it n. Keep everything symbolic.) That approach concentrates on the dynamics of the program, following the course of computations to see that they lead to the correct solution. Although that can often be made to work, we will take an alternative approach that, instead of looking at how a program changes things, concentrates on static aspects of the program: things that do not change while the program runs. We explore a method of *program verification* that allows you to give a mathematical proof that a program

works according to its specification. In addition to providing solid evidence that a program is correct, program verification has an advantage over hand simulation. Even small programs can take a computer through millions of states, as the program goes around a loop, and performing a hand simulation can be prohibitively time consuming. But the amount of work required to perform a verification by the method described here is more closely tied to the size of the program itself than to the length of the computation, so verification can be tractable when hand simulation is difficult.

24.1.1. Partial and total correctness

Program verification generally breaks down into two parts. First, you must convince yourself that any results that the program produces are correct. A program that cannot produce incorrect results is said to be *partially correct*. Partial correctness is obviously important, but it is not the whole story. A program can be partially correct, yet produce no results at all. For example, it might loop forever, never giving an answer. The second part of program verification is demonstrating that the program must eventually produce some results. A program that is both partially correct and is sure to terminate is said to be *totally correct*.

We only look at demonstrations of partial correctness here. Proofs of termination typically need to be done by other means.

24.2. Verification of flowgraphs

Flowgraphs offer a simple mechanism for expressing control, and they are a good starting point for studying program verification. The main idea is to concentrate not on the dynamics of computation, but on things that never change. There are three major steps to doing this.

1. For each edge in the flowgraph, choose a condition that is true every time the flowgraph reaches that edge. The condition associated with an edge is called an *assertion*, and it must assert properties of the *current* values of the program's variables. Remember that it is static, and does not discuss how the variables will change later, or what their values have been in the past.

2. Of course, just making a claim is not enough; the claim must be demonstrated. From the assertions that you have attached to flowgraph edges, derive *verification conditions* in a mechanical fashion so that the truth of the verification conditions implies the truth of the claim that each assertion is true whenever its edge is reached. The method of getting the verification conditions is described below.

3. The final step is to show that each of the verification conditions is true, using standard mathematical reasoning.

An example is the best place to start. Figure 24.1 shows a flowgraph with assertions attached to the edges, called an *augmented* flowgraph. The verification conditions (derived using rules from the next subsections) are as follows.

$$\text{true} \quad \Rightarrow \quad x = x \tag{24.2.1}$$

$$r = x \ \ \& \ \ r \leq y \quad \Rightarrow \quad r = \min(x, y) \tag{24.2.2}$$

$$r = x \ \ \& \ \ r > y \quad \Rightarrow \quad r = x \ \ \& \ \ r > y \tag{24.2.3}$$

$$r = x \ \ \& \ \ r > y \quad \Rightarrow \quad y = \min(x, y) \tag{24.2.4}$$

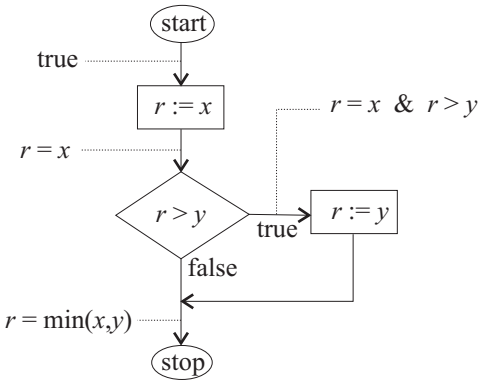


Figure 24.1: **An augmented flowgraph for computing $r = \mathbf{min}(x, y)$.** Each edge has an assertion attached to it that is a property of the variable values when the flowgraph reaches that edge. Assertion **true** indicates that nothing is known initially.

There is one verification condition for each assignment node in the flowgraph, and two for each decision node (one for the true branch and one for the false branch). Verification condition (24.2.1) expresses that the initial assignment statement, $r := x$, changes things in accordance with what the assertions say is true. Conditions (24.2.2) and (24.2.3) cover the false and true branches, respectively, of the decision node, and condition (24.2.4) is for the assignment node $r := y$.

Once you have the verification conditions, you check that each is true. You do not need to know where the conditions come from at that stage. For example, you know that $x = x$, so you know that verification condition (24.2.1) is true.

Figure 24.2 shows another example, the factorial flowgraph from Chapter 5, with an assertion attached to each edge. The assertion $n > 0$ on the start edge is a *precondition* that the input value must satisfy in order for the verification (and the program) to be correct. The remaining assertions are claimed to be true when their respective edges are reached, as long as the precondition is true at the beginning.

You should convince yourself informally that each assertion attached to an edge is true when that edge is reached during computation. Try some examples. For example, run through a computation of the factorial flowgraph with $n = 3$. Check the assertions as you go. Notice that the assertions are not assignment statements, but mathematical statements. The assertions are not part of the program. They are part of the justification that the program works.

An augmented flowgraph constitutes a verification of the flowgraph if it meets two general requirements. The first is simple: all edges pointing to a given node must have the same assertion attached to them. For example, in the augmented flowgraph for factorial, there are two edges pointing to the test $k \geq n$. Both of those edges have the same assertion attached to them. You are really attaching an assertion to the entry point for each node.

The second requirement is that verification conditions that are derived from the augmented flowgraph must be true. Rules for deriving verification conditions are described in the next two subsections.

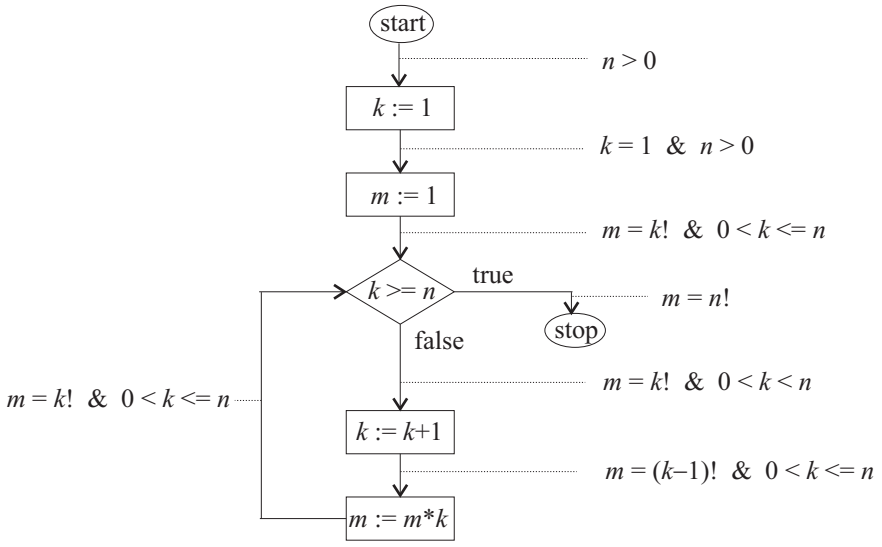
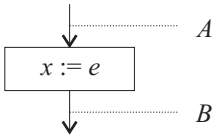


Figure 24.2: **An augmented flowgraph for computing $m = n!$.** This flowgraph contains a loop, but follows the same rules for attaching assertions to its edges as a flowgraph without a loop. Assertion $n > 0$ is a precondition that is required by this flowgraph.

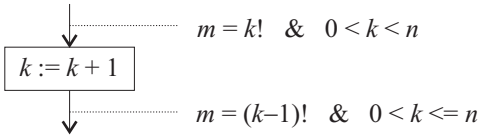
Verification conditions for assignment nodes

Suppose there is an assignment node



holding assignment $x := e$, where A is the assertion on the incoming edge and B is the assertion on the outgoing edge. (If there is more than one incoming edge, recall that all must have the same assertion on them.) Assertion A is called the *precondition* for this node, since it is presumed true before the assignment is performed, and assertion B is called the *postcondition* for the node, since it is claimed to be true after the assignment.

Let B' be the mathematical expression derived from B by replacing each occurrence of x by expression e. Then the verification condition for this assignment node is $A \Rightarrow B'$. That is, if A is true, then B' must be true. For example, consider the assignment



from Figure 24.2. Here, assertion A is $(m = k! \ \ \& \ \ 0 < k < n)$ and assertion B is $(m = (k-1)! \ \ \& \ \ 0 < k \leq n)$. Replacing each occurrence of k in assertion B by $k + 1$ (the

right-hand side of the assignment statement) yields assertion B', which is $(m = ((k+1)-1)!\ \ \&\ \ 0 < k+1 \leq n)$. So this node leads to verification condition

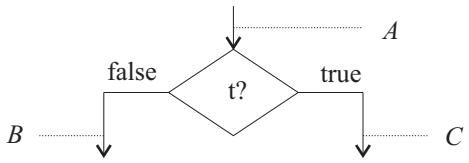
$$m = k!\ \ \&\ \ 0 < k < n \quad \Rightarrow \quad m = ((k+1)-1)!\ \ \&\ \ 0 < k+1 \leq n,$$

which is easily seen to be true for all integers k, m and n. (The term "easily" is relative. It means take a moment and work it out. You will see it. Remember that k and n are integers, so $k < n$ is equivalent to $k+1 \leq n$.)

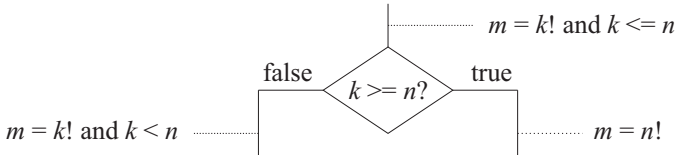
In order to understand why verification conditions for assignment statements are chosen in this way, think backwards from the postcondition to the precondition. Ask yourself, if you want assertion B to be true after performing assignment $x := v$, what assertion would have to be true before doing that assignment? Since the value of x will become v after the assignment, what you need before the assignment is assertion B', where x has been replaced by its new value v in B. For example, if you want to claim that x is prime after the assignment, you need to know that v is prime before the assignment, since v is the value that x will take on.

Verification conditions for test nodes

Suppose there is a test node



where t is the condition being tested, A is the assertion on the incoming edge, B is the assertion of the outgoing edge that is labeled false and C is the assertion on the outgoing edge that is labeled true. Two verification conditions are derived from this, and both must be proved. They are $((A\ \ \&\ \ t) \Rightarrow C)$ and $((A\ \ \&\ \ \neg t) \Rightarrow B)$, where $\neg t$ is read "not t", and is true just when t is false. The augmented factorial flowgraph, for example, contains a test node



The verification conditions are

$$m = k!\ \ \&\ \ 0 < k \leq n\ \ \&\ \ k \geq n \quad \Rightarrow \quad m = n!$$
$$m = k!\ \ \&\ \ 0 < k \leq n\ \ \&\ \ k < n \quad \Rightarrow \quad m = k!\ \ \&\ \ 0 < k < n$$

Each of these conditions is easy to check. For the first one, since $k \leq n$ and $k \geq n$, it must be the case that $k = n$. Try extracting the remaining verification conditions from the augmented flowgraph in Figure 24.2, and check that each is true.

Deriving assertions

Up to now, we have assumed that assertions have already been attached to the edges of a flowgraph. But it is reasonable to ask where they come from. Like the flowgraph itself, they are the product of the programmer's reasoning. To augment a flowgraph with assertions, begin by adding obvious facts. For example, at a test with condition t, add t to the true branch, and $\neg t$ to the false branch. For simple flowgraphs, such as the minimum-calculator in Figure 24.1, that might be all you need. In fact, you might find that some parts of the assertions are unnecessary, and you can remove them.

The presence of loops often requires more thought. Ask yourself what the loop is accomplishing, and what is true each time you reach a particular node in the loop. If you do a hand simulation of the flowgraph of Figure 24.2 with $n = 4$, and look at the values of n, k and m each time you reach the choice node, you get the following collections of values.

$$\{k = 1, m = 1, n = 4\}$$
$$\{k = 2, m = 2, n = 4\}$$
$$\{k = 3, m = 6, n = 4\}$$
$$\{k = 4, m = 24, n = 4\}$$

Since your intent is to compute factorials, it should jump out at you that $m = k!$. The remaining assertions are added to make the verification condions provable, by trial and error and some thought.

24.3. Verification of structured programs

Every structured program can be converted to an equivalent flowgraph. So one approach to verification of structured programs is to convert to a flowgraph and perform the verification on the flowgraph. It is possible, however, to perform a verification of a structured program without conversion to a flowgraph. Instead of writing the entire program and then attaching assertions, you derive the assertions and the program together, building up larger pieces from smaller ones. So you build *verified* programs rather than just plain programs.

A *Hoare triple* has the form $\{A\}\, S\, \{B\}$ where A and B are assertions and S is a statement or sequence of statements. The meaning of that triple is that, if A is true, and then statement(s) S are performed, then B will be true afterwards. For example, Hoare triple $\{k = 1\}\ k := k + 1\ \{k = 2\}$ says that, if assertion $k = 1$ is true now, and statement $k := k + 1$ is performed, then assertion $k = 2$ will be true afterwards. In Hoare triple $\{A\}\, S\, \{B\}$, assertion A is called the *precondition* and assertion B is called the *postcondition.*

Verification is done by using rules for inferring true Hoare triples from other true Hoare triples using a system of reasoning is called *Hoare logic*, named after its inventor. Rules of inference are written $\dfrac{a}{b}$ where a is a collection of known facts and b is a fact that is derived from them. If there is nothing above the line, then what is below the line can be concluded without any prior facts.

Figure 24.3 shows rules of inference for the Hoare logic of assignments, sequencing, if-then-else and while-do, the same tiny language whose operational semantics is defined in Chapter 22, omitting the if-statement without an else part. We assume that expressions only compute values, and do not change anything. Notation $A[x \leftarrow e]$ indicates the expression obtained by replacing every occurrence of variable x in expression A by expression e. For example, if A is $x + x$, then $A[x \leftarrow 9]$ is 9+9. Parentheses need to be used if necessary to

Rule Name	Rule
(Hoare-assign)	$\overline{\{A[x \leftarrow e]\}\ x\ :=\ e;\ \{A\}}$
(Hoare-sequence)	$\dfrac{\{A\}\,S\,\{B\},\ \{B\}\,T\,\{C\}}{\{A\}\ S\,T\ \{C\}}$
(Hoare-if)	$\dfrac{\{A\ \&\ C\}\,S\,\{B\},\ \{A\ \&\ \neg C\}\,T\,\{B\}}{\{A\}\ \textbf{if}\ C\ \textbf{then}\ S\ \textbf{else}\ T\ \textbf{endif}\ \{B\}}$
(Hoare-while)	$\dfrac{\{A\ \&\ C\}\,S\,\{A\}}{\{A\}\ \textbf{while}\ C\ \textbf{do}\ S\ \textbf{endwhile}\ \{A\ \&\ \neg C\}}$
(Hoare-weaken)	$\dfrac{A \Rightarrow B,\ C \Rightarrow D,\ \{B\}\,S\,\{C\}}{\{A\}\,S\,\{D\}}$

Figure 24.3: **Inference rules for Hoare logic.**

prevent operator precedence from changing how the expression is broken up into parts. For example, if A is $x * y$, then $A[x \leftarrow u + v] = (u + v) * y$, not $u + v * y$. Notation $\neg A$ indicates that A is false.

Most of the Hoare rules are easy to justify. The sequence rule (Hoare-sequence), for example, presumes that you already know $\{A\}\,S\,\{B\}$ and $\{B\}\,T\,\{C\}$. Since the postcondition for statement S is identical to the precondition for statement T, those two triples can be combined to conclude that $\{A\}\,S\,T\,\{C\}$. Rule (Hoare-weaken) states that you can always strengthen the precondition and weaken the postcondition. For example, if $\{k \geq 0\}\,S\,\{n > 0\}$ is known to be true, then certainly $\{k > 0\}\,S\,\{n \geq 0\}$ must also be true.

Rule (Hoare-while) for the while loop needs some discussion. Assertion A is called an *invariant* of the loop, and is an assertion that is true each time the loop reaches its top. Suppose that, in a given program, A is an invariant for a given loop **while** C **do** S **endwhile**. Then A will certainly need to be true when the loop starts, since otherwise it would not be true every time the loop is at its top. That is, A is a precondition of the loop.

Now think about the last time the loop reaches its top, just before the loop ends. Since A is an invariant, A will be true. But if the loop is to exit, then C must be false. The program goes to the end of the loop, with A true and C false. Putting this together, we get Hoare triple $\{A\}$ **while** C **do** S **endwhile** $\{A\ \&\ \neg C\}$. But this presumes that A is indeed an invariant of the loop. In order for that to be true, statement(s) S must not ruin the invariant. Notice that S is only performed when C is true; to show that S preserves the truth of A, we need to show that, if A and C are both true when S starts, then A is true again when S is finished. So what is required is $\{A\ \&\ C\}\,S\,\{A\}$.

A short sample verification

To prove characteristics of programs using Hoare-style rules, you need to keep two things in mind. First, build up the verification as you build up the program. So start with small pieces. Second, keep your objective in mind. That is, prove what you want to prove, not just what is obviously true. Suppose that you want to prove that program fragment

if $x > y$ **then**
  $r := x$;
else
  $r := y$;
end if

sets r to $\max(x, y)$, the larger of x and y. Start with the two assignment statements. But remember that you are only interested in what statement $r := x$ does in the case when $x > y$, because that is the only time it is used. Obviously, it sets $r = x$. But, more to the point, it sets $r = \max(x, y)$ in the case where it is used. Using (Hoare-assign) twice, derive the following two Hoare-triples.

$$\{x = \max(x, y) \ \ \& \ \ x > y\} \quad r := x; \quad \{r = \max(x, y) \ \ \& \ \ x > y\}$$
$$\{y = \max(x, y) \ \ \& \ \ \neg(x > y)\} \quad r := y; \quad \{r = \max(x, y) \ \ \& \ \ \neg(x > y)\}$$

It was necessary to put $x > y$ or $\neg(x > y)$ in both the precondition and the postcondition of each Hoare-triple in order for rule (Hoare-assign) to apply. But you do not really care whether $x > y$ or not after the assignment is finished, so remove that irrelevant information. Also, notice that $x > y \Rightarrow x = \max(x, y)$ and $\neg(x > y) \Rightarrow y = \max(x, y)$, so the preconditions can be simplified. Use rule (Hoare-weaken) to derive the following.

$$\{\text{true} \ \ \& \ \ x > y\} \quad r := x; \quad \{r = \max(x, y)\}$$
$$\{\text{true} \ \ \& \ \ \neg(x > y)\} \quad r := y; \quad \{r = \max(x, y)\}$$

Now use rule (Hoare-if), where C is $x > y$.

$$\{\text{true}\} \quad \textbf{if } x > y \textbf{ then } r := x \textbf{ else } r := y \textbf{ end if} \quad \{r = \max(x, y)\}$$

A longer sample verification

It is possible to demonstrate from the precondition $n > 0$ that the following program ends satisfying postcondition $m = n!$.

$k := 1$;
$m := 1$;
while $k < n$ **do**
  $k := k + 1$;
  $m := m * k$;
endwhile

The proof goes as follows, carried out in detail. Hoare triples are broken into lines instead of being written in a single line. You should see strong similarities between this proof and the flowgraph verification. Check the steps.

Step	Conclusion	Justification
1	$\{1 = 1 \ \ \& \ \ n > 0\}$ $k := 1$; $\{k = 1 \ \ \& \ \ n > 0\}$	by (Hoare-assign)
2	$\{n > 0\}$ $k := 1$; $\{k = 1 \ \ \& \ \ n > 0\}$	by fact (1) and rule (Hoare-weaken). This is because $(n > 0 \Rightarrow 1 = 1 \ \ \& \ \ n > 0)$. We will not continue to state such obvious facts.

3	$\{k = 1 \ \& \ 1 = 1! \ \& \ n > 0\}$ $m := 1;$ $\{k = 1 \ \& \ m = 1! \ \& \ n > 0\}$	by (Hoare-assign)
4	$\{k = 1 \ \& \ n > 0\}$ $m := 1;$ $\{m = k! \ \& \ 0 < k \leq n\}$	by fact (3) and rule (Hoare-weaken), and obvious facts about the integers. For example, $(k = 1$ and $n > 0) \Rightarrow k \leq n$ when k and n are integers.
5	$\{n > 0\}$ $k := 1;$ $m := 1;$ $\{m = k! \ \& \ 0 < k \leq n\}$	by facts (2) and (4) and rule (Hoare-sequence)
6	$\{m = ((k + 1) - 1)! \ \& \ 0 < k + 1 \leq n\}$ $k := k + 1;$ $\{m = (k - 1)! \ \& \ 0 < k \leq n\}$	by (Hoare-assign)
7	$\{m = k! \ \& \ 0 < k < n\}$ $k := k + 1;$ $\{m = (k - 1)! \ \& \ 0 < k \leq n\}$	by fact (6) and rule (Hoare-weaken), and obvious facts about the integers
8	$\{m * k = k! \ \& \ 0 < k \leq n\}$ $m := m * k;$ $\{m = k! \ \& \ 0 < k \leq n\}$	by (Hoare-assign)
9	$\{m = (k - 1)! \ \& \ 0 < k \leq n\}$ $m := m * k;$ $\{m = k! \ \& \ 0 < k \leq n\}$	by fact (8) and rule (Hoare-weaken), and the fact that $m = (k - 1)! \Rightarrow m * k = k!$ when $k > 0$
10	$\{m = k! \ \& \ 0 < k < n\}$ $k := k + 1;$ $m := m * k;$ $\{m = k! \ \& \ 0 < k \leq n\}$	by facts (7) and (9) and rule (Hoare-sequence)
11	$\{m = k! \ \& \ 0 < k \leq n \ \& \ k < n\}$ $k := k + 1;$ $m := m * k;$ $\{m = k! \ \& \ 0 < k \leq n\}$	by fact (10) and rule (Hoare-weaken)

12	$\{m = k! \ \& \ 0 < k \leq n\}$
	while $k < n$ **do**
	$k := k + 1;$
	$m := m * k;$
	endwhile
	$\{m = k! \ \& \ 0 < k \leq n \ \& \ \neg(k < n)\}$
		by fact (11) and rule (Hoare-while)

13	$\{m = k! \ \& \ 0 < k \leq n\}$
	while $k < n$ **do**
	$k := k + 1;$
	$m := m * k;$
	endwhile
	$\{m = n!\}$
		by fact (12) and rule (Hoare-weaken), and obvious facts about the integers. Notice that $(k \leq n) \ \& \ \neg(k < n) \Rightarrow k = n$. But then, if $m = k!$, surely $m = n!$.

14	$\{n > 0\}$
	$k := 1;$
	$m := 1;$
	while $k < n$ **do**
	$k := k + 1;$
	$m := m * k;$
	endwhile
	$\{m = n!\}$
		by facts (5) and (13) and rule (Hoare-sequence)

24.4. Axiomatic semantics

A program is only useful in practice to the extent that the programmer understands what it does, and can convince himself or herself of the program's correctness. If a program were written, for example, by randomly typing on the keyboard, and nobody understood the program, then the program would almost surely need to be thrown away, even if, by some fantastic fortune, it happened to do exactly what was desired. Indeed, many a working computer program or procedure definition, written by another programmer, has been thrown away because a current programmer did not understand it, and preferred to substitute his or her own version that was more easily understood (by him or her).

It might be argued, therefore, that the semantics of a programming language is only useful to the extent that it helps a programmer reason about the behavior of programs. Suppose that a programmer has decided to use Hoare logic to reason about programs. Then the inference rules in Figure 24.3 are the only facts about programs that the programmer will use. They might just as well *be* the semantics of programs.

A semantics based on providing facts and rules of inference for reasoning about programs is called an *axiomatic semantics*. The Hoare-logic rules of inference form an axiomatic semantics of a simple programming language. Rather than discussing what a program does when it is run, the axiomatic semantics only tells you which facts can be proved about programs. Axiomatic semantics have the advantage that they can tell a programmer important facts about a programming language without becoming mired in the details of how computation is performed. They work at a level of abstraction above all of those details, and so are a very high level semantics.

By the same token, however, by suppressing the details, an axiomatic semantics can leave a programmer uneasy about what a program does, and how it works. A practical approach to providing the semantics of a programming language is to provide multiple consistent semantics. An operational semantics might be useful to a programmer for some parts of the programming process. At other stages, the programmer might prefer to think in terms of an axiomatic semantics that is consistent with the operational semantics, but that suppresses unwanted details. Providing more than one form of semantics is probably the most reasonable approach to defining a programming language.

24.5. Practical reasoning about programs

The goal of program verification is to prove, beyond any doubt, that a program performs according to some clearly stated requirements. It should be fairly evident from the examples that full verification of programs is a very lengthy and difficult process. Indeed, there can be so many steps that, in any verification performed by a human, there will almost surely

be some mistakes. The potential presence of those mistakes casts doubt on the verification itself, and hence on the software that has purportedly been verified. It can be argued that a further verification of the verification is required! Does this apparent circularity spell the death of program verification? Fortunately, there are ways out of the difficulties.

Informal verification

One way out is to relax the goal. In real software design, it might not be necessary to have absolute faith that there are no mistakes in a program. Nonetheless, any method that will increase confidence in the program's correctness and will decrease the presence of mistakes will be useful. Program verification offers just such a method. A programmer should think through at least a sketch of a proof that his or her program works according to specifications. Hoare logic provides a framework for such proof sketches.

Difficulties in imperative programs are often associated with loops. Hoare logic suggests that you find a loop invariant for your loop, not just after you have written it, but during the process of writing the loop. For example, suppose that you need a function to compute x^n, where x is a number and n is a positive integer. You can compute successive powers x^1, x^2, x^3, ..., x^n. Those powers will need to be stored somewhere, so suppose that variable r successively holds those powers. Somehow, the program must keep track of which of those powers it currently has. Let variable k play that role. Then a loop invariant will be $r = x^k$. The program must begin by performing some assignment statements that make this invariant true. (It suffices to set $r := x$ and $k := 1$.) Then it must perform a loop, each time making k larger so that the desired power x^n will be approached, and each time keeping the invariant true. Statements $k := k + 1; r := r * x;$ will perform the updates. A little thought shows that those steps keep assertion $r = x^k$ true. Now when should the loop stop? Since $r = x^k$ throughout the loop, and we want $r = x^n$ when the loop stops, the loop should obviously stop when $k = n$. This reasoning leads to the following program fragment that sets $r = x^n$.

```
r := x;
k := 1;
while k ≠ n do
   k := k + 1;
   r := r * x;
endwhile
```

Although it has not been fully verified, a rough verification of the program fragment, based on the idea of a loop invariant, has been done during the process of writing it.

Formal verification

Complete verification of a large program by a human is out of reach. No person will be able to do the entire job with such care that even he or she will be confident that there are no errors in his or her own work; certainly, nobody else will have such confidence. Formal verification can be done, though, provided the verification is checked by a computer. Automatic proof checkers can be used to check that verifications have been done correctly. The best proof checkers not only check the steps of a proof, but can also fill in some missing details, freeing the human from having to write down each and every tiny step of the proof.

24.6. Demonstrating correctness of equational programs

For equational programming, the equations that you write down *are* the verification conditions, and their correctness guarantees partial correctness of your program. You need to settle on a clear and precise semantic definition of each function, and then show that your equations are true, according to that definition. A program that only contains correct equations will never produce an incorrect answer.

Proving termination for elementary recursive definitions

That leaves the issue of termination. It is possible to write equations that are true, but that cannot be used to produce any answers. An obvious one is to define function f by equation $f(x) = f(x)$, an equation that is certainly is true, but that leads to an infinite recursion when it is used.

Termination can be a sticky issue in a setting where lazy evaluation is employed and data structures can be infinitely large, as described in Chapter 10. For simplicity, we assume that all lists are finitely long here, and we will concentrate on computations that will terminate regardless of whether evaluation is lazy or eager.

We will start by restricting attention to function definitions that can employ recursion, but that do not use mutual recursion among several functions. Then, to demonstrate termination, it suffices show that each recursive call uses a parameter that is, in some (suitable) sense, smaller than the one you started with. If $f(x) = e$ is one of your equations, where e is an expression, then you show that each call to f inside e uses a parameter whose size is smaller than the size of x. You also need to ensure that, whatever measure of size you are using, it is not possible to do infinitely many steps of decreasing the size. Typically, the size is a nonnegative integer, and, since you cannot decrease it below 0, it cannot continue to be decreased forever. A common notion of size is the length of a list. In definition

$$\begin{aligned} \text{length}\,[\,] &= 0 \\ \text{length}(x :: xs) &= 1 + \text{length}\ xs \end{aligned}$$

the recursive call uses a list xs that is shorter than x::xs. When the parameter of a function is an ordered tuple, you often choose the size of one of the tuple's members to be the size of the tuple. For example, suppose that function drop is defined as follows.

$$\begin{aligned} \text{drop}(0, x) &= x \\ \text{drop}(n, [\,]) &= [\,] \\ \text{drop}(n+1, x :: xs) &= \text{drop}(n, xs) \end{aligned}$$

Of those equations, only the last one involves recursion. The parameter of drop is an ordered pair; suppose the size of pair (n, x) is chosen to be the length of list x. Notice that pair (n, xs) is smaller, in that sense, than pair $(n+1, x :: xs)$, so, since the size is a nonnegative integer, the recursion must eventually terminate. For the definition of drop, more than one notion of size will work. If you define the size of (n, x) to be n, then (n, xs) is smaller than $(n+1, x :: xs)$, and again you have argued that the recursion must terminate.

Not every notion of size will work. For example, if you choose a size that is an integer (positive or negative) then you can continue decreasing the size forever, from -1 to -2, etc. But there is a concept that captures just what is required of a size. A set S of values, along with an ordering relation on S, is called *well-founded* if it is not possible to produce an

infinite decreasing sequence $x_1 > x_2 > x_3 > \ldots$ of members of set S. The most commonly used well-founded set is the set of nonnegative integers, with the usual ordering, but there are others. The set of all strings, ordered alphabetically, is well-founded as long as you define alphabetical ordering so that $s < t$ when s is a prefix of t. Another example of a well-founded set is the set of ordered pairs of nonnegative integers, ordered *lexicographically*, where you say that $(x, y) < (u, v)$ if either $x < u$ or $x = u$ and $y < v$. For example, the following equations define a function called Ackermann's function.[1] Values m and n are nonnegative integers, and A produces a nonnegative integer.

$$\begin{aligned} A(0, m) &= m + 1 \\ A(n + 1, 0) &= A(n, 1) \\ A(n + 1, m + 1) &= A(n, A(n + 1, m)) \end{aligned}$$

Let pairs (x, y) be ordered lexicographically. The second equation starts with pair $(n + 1, 0)$ and does a recursive call with pair $(n, 1)$, which is smaller than pair $(n + 1, 0)$ in the lexicographic order. The third equation makes two recursive calls. But $(n + 1, m) < (n + 1, m + 1)$ in lexicographic ordering, and $(n, z) < (n + 1, m + 1)$ regardless of what z is, so both recursive calls to A use smaller parameters.

Mutual recursion

A definition is mutually recursive if there are two or more functions that use one another; you might find, for example, that f_1 uses f_2 and f_2 uses f_1. To prove that a mutually recursive definition must terminate, assign each function an integer value in any way that you like. Choose a single well-founded ordering on pairs (n,x) where n is a function number and x is a value of the type that the n-th function takes. (So the type of x can depend on the value of n.) Suppose that equation $f(x) = e$ uses $g(y)$ inside expression e. If f is numbered n_f and g is numbered n_g, then you show that pair $(n_g, y) < (n_f, x)$ in your selected well-founded ordering.

For example, suppose that trees are defined so that each tree is either a leaf containing an integer or is a node with a list of zero or more subtrees. Let's write leaf(n) to indicate a leaf holding integer n, and node(ts) to indicate a node with list of subtrees ts. Function numleaves(t) returns the number of leaves in tree t, and totalleaves(ts) takes a list ts of trees, and returns the sum, over all trees t in ts, of the number of leaves in t.

$$\begin{aligned} \text{numleaves}(\text{leaf}(n)) &= 1 \\ \text{numleaves}(\text{node}(ts)) &= \text{totalleaves}(ts) \\ \text{totalleaves}([\,]) &= 0 \\ \text{totalleaves}(t :: ts) &= \text{numleaves}(t) + \text{totalleaves}(ts) \end{aligned}$$

Assume that lists and trees are required to be finitely large. Let nodes(t) be the number of nodes and leaves in tree t, and nodes(L) be the sum of nodes(t) over all trees t in list-of-trees L. Let numleaves be function 1 and totalleaves be function 2. Say that $(n_1, a) < (n_2, b)$ just when nodes(a) < nodes(b), where a and b are either lists or trees. You can see that all of the function calls use smaller value in this sense since nodes(node(ts)) < nodes(ts) and nodes(t) < nodes($t :: ts$) and nodes(ts) < nodes($t :: ts$).

[1] Be careful with this innocent looking definition. Ackermann's function grows very rapidly, and takes a very long time to compute. Although A(10,10) must terminate, it will probably not do so in your lifetime, even on the fastest computer.

24.7. Proving characteristics of functions

Each function in a program should really have two definitions, the semantic definition that tells what you want the function to compute and the program that tells how the function is computed. Proving that a function definition is partially correct amounts to proving that the two definitions are consistent with one another.

Equational programs are often so close to a semantic definition that it is difficult to come up with any better definition than the program itself. The length of a list, for example, is difficult to define more concisely than to say that length($[\,]$) = 0 and length($x :: xs$) = $1 + \text{length}(xs)$. In those cases, you do not have two definitions to check against one another, and you just have to accept the equational definition as correct.

But there are still some assertions that you might want to prove, involving properties that you believe your functions possess. Suppose that you have defined concatenation of lists, as follows.

$$[\,] ++ y = y \tag{24.7.1}$$
$$(h :: t) ++ y = h :: (t ++ y) \tag{24.7.2}$$

One property that ++ possesses is associativity:

$$(x ++ y) ++ z = x ++ (y ++ z). \tag{24.7.3}$$

You can prove Equation (24.7.3) directly from the definition of ++ using the notion of *mathematical induction*. Do a thought experiment where you prove the equation for various lists x (and arbitrary y and z) starting with small lists x.

Step 0. Prove Equation (24.7.3) for $x = [\,]$ (and all y and z).
Step 1. Prove Equation (24.7.3) for all lists x that have length 1 (and all y and z).
Step 2. Prove Equation (24.7.3) for all lists x that have length 2 (and all y and z).

This process has to keep going forever, proving Equation (24.7.3) for larger and larger lists x. But it does not really matter that you never finish, because this is only a thought experiment. All that you really need to do is to show that, for an arbitrary n, you can carry out Step n. But notice that, when you are doing that step, you have already completed all of the prior steps. So you can assume that Equation (24.7.3) is true for lists x that have fewer than n members; you proved that earlier. Here are the arguments, consisting of step 0 and step n for an arbitrary $n > 0$.

Step 0.

$$\begin{aligned} ([\,] ++ y) ++ z &= y ++ z && \text{by (24.7.1)} \\ [\,] ++ (y ++ z) &= y ++ z && \text{by (24.7.1)} \end{aligned}$$

Since the two left-hand sides are equal to the same thing, they are equal to one another, so Equation 24.7.3 is true in the case where $x = [\,]$.

Step n. If list x has length $n > 0$, then x must have the form $h :: t$ for some h and t. But t is shorter than x, so we have already proved, at step $n - 1$, that

$$(t ++ y) ++ z = t ++ (y ++ z). \tag{24.7.4}$$

Now use Equations (24.7.2) and (24.7.4) to find expressions equivalent each of $(x +\!\!+ y) +\!\!+ z$ and $x +\!\!+ (y +\!\!+ z)$.

$$\begin{array}{rcll}
(x +\!\!+ y) +\!\!+ z & = & ((h :: t) +\!\!+ y) +\!\!+ z & \text{since } x = h :: t \\
& = & (h :: (t +\!\!+ y)) +\!\!+ z & \text{by (24.7.2)} \\
& = & h :: ((t +\!\!+ y) +\!\!+ z) & \text{by (24.7.2)} \\
& = & h :: (t +\!\!+ (y +\!\!+ z)) & \text{by (24.7.4)} \\
x +\!\!+ (y +\!\!+ z) & = & (h :: t) +\!\!+ (y +\!\!+ z) & \text{since } x = h :: t \\
& = & h :: (t +\!\!+ (y +\!\!+ z) & \text{by (24.7.2)}
\end{array}$$

Since both are equal to the same thing, we have proved that Equation 24.7.3 is true for lists x of length n, and for arbitrary lists y and z. ∎

Example: length$(x +\!\!+ y)$ = length(x) + length(y)

The length function is defined by

$$\begin{array}{rcll}
\text{length}([\,]) & = & 0 & (24.7.5) \\
\text{length}(h :: t) & = & 1 + \text{length}(t) & (24.7.6)
\end{array}$$

Here is a proof, from the defining equations of length and ++, that length(x) + length(y) = length$(x +\!\!+ y)$, using mathematical induction on list x.

Step 0. Suppose that $x = [\,]$.

$$\begin{array}{rcll}
\text{length}([\,]) + \text{length}(y) & = & 0 + \text{length}(y) & \text{by (24.7.5)} \\
& = & \text{length}(y) & \\
\text{length}([\,] +\!\!+ y) & = & \text{length}(y) & \text{by (24.7.1)}
\end{array}$$

so length(x) + length(y) = length$(x +\!\!+ y)$ in the case where $x = [\,]$.

Step n. If list x has length $n > 0$, then x must have the form $h :: t$ for some h and t. But t is shorter then x, so we have already proved, at step $n - 1$, that

$$\text{length}(t) + \text{length}(y) = \text{length}(t +\!\!+ y). \qquad (24.7.7)$$

Now

$$\begin{array}{rcll}
\text{length}(x) + \text{length}(y) & = & \text{length}(h :: t) + \text{length}(y) & \text{since } x = h :: t \\
& = & (1 + \text{length}(t)) + \text{length}(y) & \text{by (24.7.6)} \\
& = & 1 + (\text{length}(t) + \text{length}(y)) & \\
& = & 1 + \text{length}(t +\!\!+ y) & \text{by (24.7.7)} \\
\text{length}(x +\!\!+ y) & = & \text{length}((h :: t) +\!\!+ y) & \text{since } x = h :: t \\
& = & \text{length}(h :: (t +\!\!+ y)) & \text{by (24.7.2)} \\
& = & 1 + \text{length}(t +\!\!+ y) & \text{by (24.7.6)}
\end{array}$$

so length(x) + length(y) = length$(x +\!\!+ y)$ in this case too. ∎

24.8. Exercises

24.1. Which of the following are reasonable assertions that might be attached to an edge of a flowgraph? An assertion that is always false is not a reasonable one.

(a) $x > y$

(b) $x = x + 1$

(c) Variable x increases by one each time around the loop.

(d) Variable x has been converted into meters.

(e) The new x is twice as large as the old x.

(f) Variable x holds a nonempty string that begins with letter 'a'.

24.2. Suppose that you would like assertion $x > y + 10$ to be true after performing assignment statement $x := 2 * y - 17$. What precondition for the assignment will guarantee that postcondition? Give the most general possible precondition. For example, precondition $y = 100$ guarantees the postcondition, but it is too restrictive, and excludes other values that will also work, so it is not the most general precondition. (Hint: Use the assignment rule (Hoare-assign). It naturally works backwards.)

24.3. Suppose that you would like assertion $x > y + 10$ to be true after performing sequence of statements $y := y + z; x := 2 * y - 17; x := x + 1$. What is the most general precondition for this sequence of statements that will guarantee that postcondition? (Hint: work backwards across the sequence of statements, using rule (Hoare-assign).)

24.4. An empty assertion in a Hoare triple is implicitly true. Prove that Hoare triple {} x := 0 {x = 0} is true, using the rules in Figure 24.3.

24.5. Write all of the verification conditions for the augmented flowgraph in Figure 24.2, and check that each verification condition is true.

24.6. Use the power program fragment of Section 24.5 as a guide for this problem

(a) Draw a flowgraph for a program that computes $p = x^n$, where x and n are positive integers that are the inputs to the program.

(b) Determine assertions to attach to the edges, and show them in the flowgraph. Try a hand simulation as a check that your assertions are reasonable. Is each one true each time you reach its edge?

(c) Derive all verification conditions for your augmented flowgraph.

(d) Argue that each verification condition true. Be sure not to refer back to the flowgraph in this step. You are only concerned with the verification conditions.

24.7. Some languages include a **null** statement that does nothing. Give a Hoare-style rule of inference for a null statement.

24.8. Figure 24.3 shows a rule for an if-then-else statement, but does not include a rule for an if-then statement, without an else part. Give a Hoare-style rule of inference for an if-then statement.

24.9. Give a Hoare-style inference rule for the C or Java do-loop, which has the test at the end instead of at the beginning. It performs the body once before performing the test to see whether to continue the loop.

24.10. What function f do Equations (24.8.1) define? Express f in "closed form", as a single non-recursive equation. Assume that f only applies to nonnegative integers. Prove that your assertion about f is true. (**Hint.** Try computing $f(x)$ for a few small integer values of x, and look for a pattern.)

$$\begin{array}{rcl} f(0) & = & 0 \\ f(n+1) & = & f(n) + 2n + 1 \end{array} \tag{24.8.1}$$

24.11. What function s do Equations (9.6.1) define? Assume that s only applies to integers. Prove your assertion first for nonnegative x, and then prove it for negative x.

24.12. Let f be an arbitrary function, and define function mapf as follows.

$$\begin{array}{rcl} \text{mapf}([\,]) & = & [\,] \\ \text{mapf}(h :: t) & = & f(h) :: \text{mapf}(t) \end{array}$$

(a) Expressed in terms of f, a, b and c, what list does mapf([a,b,c]) produce?

(b) Assume that $f(z)$ produces an answer for every z. From the definitions of mapf and ++, prove that mapf(x ++ y) = mapf(x) ++ mapf(y) for all lists x and y. This equation is true regardless of what function f computes.

24.13. Using Equations (24.7.5) and (24.7.6) and the definition of mapf from the preceding question, prove that length(mapf(x)) = length(x).

24.14. Prove that the set of ordered pairs of integers, ordered lexicographically, is well-founded.

24.15. Suppose that a binary tree has two variants, nil (an empty tree) and node(x, L, R), where x is the node's label, L is the left subtree and R is the right subtree. Function mirror is defined as follows.

```
mirror(nil)           = nil
mirror(node(x, L, R)) = node(x, mirror(R), mirror(L))
```

Prove that mirror(mirror(t)) = t for all binary trees t.

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