The Symbols of Logic

We've already seen "∴" used to mean "therefore," signifying the conclusion of our argument.

It is also customary to express:

- Propositions with capital letters: P, Q, R, ...
- Propositional variables with lowercase letters in italics: x, y, z, ...
- Predicates as functions: P(x), R(y, z), ...
- Implications with arrows: $P \rightarrow Q$
- Disjunctions with this symbol: "$P \vee Q$"
- Conjunctions with this symbol: "$P \wedge Q$"
- Negation with this symbol: "$\neg P$"
- "For all" with "$\forall$" and "there exists" with "$\exists$"

Getting English from Symbols

Symbol	Symbol
P	29 is a prime number
Q	29 is the sum of two squares
R	29 leaves a remainder of 1 when divided by 4
S	There is a perfect square which is 1 less than a multiple of 29

1. $P \wedge Q$
2. $R \rightarrow S$
3. $(R \wedge P) \rightarrow Q$
4. $R \rightarrow (P \vee \neg S)$
5. $P \rightarrow [(Q \rightarrow S) \wedge (S \rightarrow Q)]$

Getting English from Symbols

Symbol	Meaning
E(x)	Bob eats x
M(s, t)	s has seen t
S(t)	Beth has seen t
T(a, b)	a and b are sides of the same triangle
L(a, b)	a and b have the same length
B(a, t)	a belongs to t

1. E(tomatoes)

2. $\forall x$ E(x)

3. M(Rachel, *The Princess Bride*)

4. $\forall x$ M(Rachel, x)

Getting English from Symbols

5. $\forall t\ (S(t) \rightarrow M(\text{Betty}, t))$

6. $\exists h\ \neg S(h)$

7. $T(a, b) \rightarrow L(a, b)$

8. $(B(x, y) \wedge T(w, x)) \rightarrow B(w, y)$

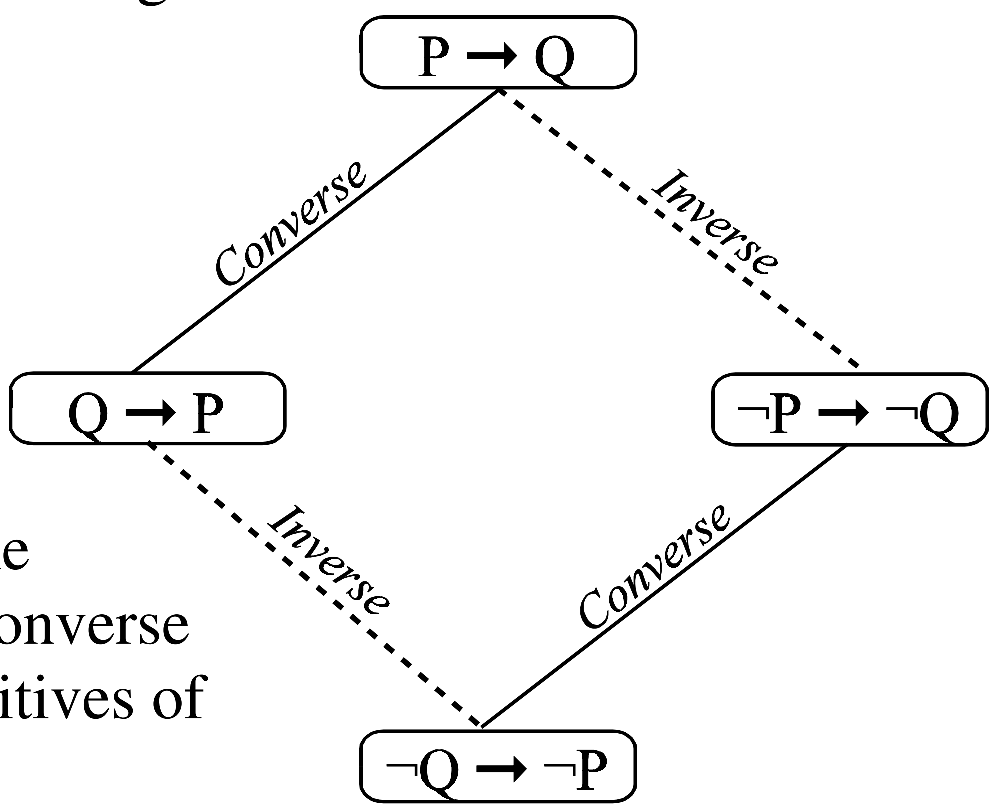
9. $\forall a\ \forall b\ \forall c\ [(T(a, b) \wedge T(b, c)) \rightarrow T(a, c)]$

Implications

Given an implication, P → Q, we can form three variants:

Original	P → Q
Converse	Q → P
Inverse	¬P → ¬Q
Contrapositive	¬Q → ¬P

These can be arranged as follows:



Notice that the inverse and converse are contrapositives of one another

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Implication Variations

Find the inverse, converse and contrapositive of each of these implications:

- If it rains, then I will not go to the beach

Converse	
Inverse	
Contrapositive	

- I play well if I get enough sleep

Inverse	
Converse	
Contrapositive	

Compound Propositions

A compound proposition is one that is built from simpler propositions using the connectives $\wedge$, $\vee$, $\rightarrow$ and $\neg$. (There are others, but they don't concern us here.)

Some sample compound propositions:

1. $P \wedge Q$

2. $R \rightarrow S$

3. $(R \wedge P) \rightarrow Q$

4. $R \rightarrow (P \vee \neg S)$

5. $P \rightarrow [(Q \rightarrow S) \wedge (S \rightarrow Q)]$

Note that the truth value of a compound proposition can be determined from the truth values of its simple components.

Compound Propositions

Suppose I tell you that:

P is True, Q is False, R is True, S is False

What is the value of each of the following:

1. $P \wedge Q$

2. $R \rightarrow S$

3. $(R \wedge P) \rightarrow Q$

4. $R \rightarrow (P \vee \neg S)$

5. $P \rightarrow [(Q \rightarrow S) \wedge (S \rightarrow Q)]$

Truth Tables — Tedious but Foolproof

Given a compound proposition with *k* simple components we can build the truth table which has a column for each simple proposition, a column for the compound proposition (and perhaps some other columns to help with the construction) and a row for each *k*-tuple of truth values that the simple propositions can have. (That's 2^k rows altogether.)

P	Q	R	$(R \wedge P)$	$(R \wedge P) \rightarrow Q$

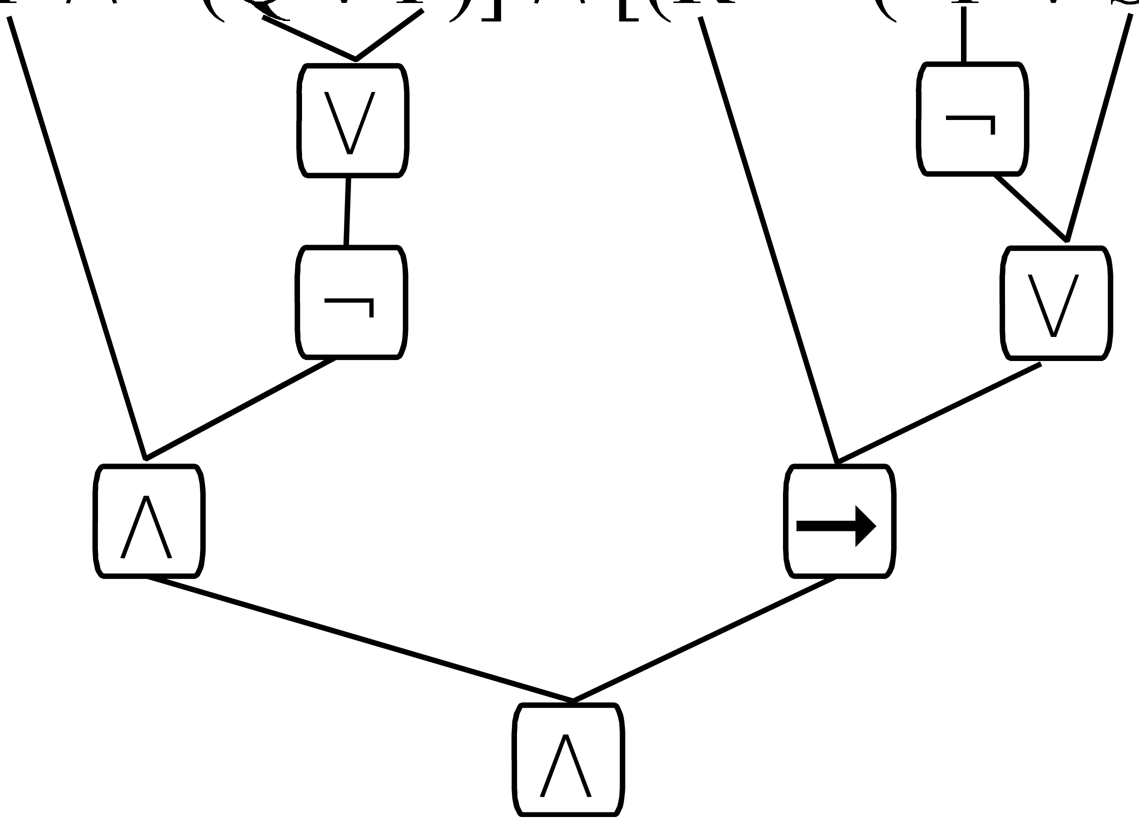
Truth Tables

We can build a truth table for any comound proposition, no matter how complex, as long as we know how to build the truth tables for the simplest compound propositions:

$P \vee Q$, $P \wedge Q$, $P \rightarrow Q$, and $\neg P$

All other compound expressions are just repeated applications of these:

$$[P \wedge \neg(Q \vee P)] \wedge [(R \rightarrow (\neg P \vee S)]$$



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Simplest Truth Tables

Here are the simplest truth tables.

P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	F

P	Q	$P \vee Q$
T	T	T
T	F	T
F	T	T
F	F	F

P	Q	$P \rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

P	$\neg P$
T	F
F	T

Your Turn

Construct truth tables for the following compound propositions:

- $\neg Q \rightarrow \neg P$

- $(R \lor P) \land (\neg R \lor Q)$

Logical Equivalence

Two compound propositions are said to be *logically equivalent* if they have the same truth values regardless of the truth values of the underlying simpler propositions.

We use the symbol "≡" to denote logical equivalence.

Examples:

$\neg Q \rightarrow \neg P \equiv P \rightarrow Q$

$(R \lor P) \land (\neg R \lor Q) \equiv P \lor Q$

$\neg(P \land Q) \equiv \neg P \lor Q$

$\neg(P \lor Q) \equiv \neg P \land \neg Q$

(These last two are called "DeMorgan's rules," and are generally helpful for negating compound propositions.)

Negating Compound Propositions

It is sometimes useful to be able to negate a compound proposition. For example, on one homework problem we were asked to prove $\neg(A \rightarrow E)$, but it was easier to prove if we rewrote the negation as $(A \wedge \neg E)$.

For example, suppose we wish to prove $\neg P$, and we know that $P \rightarrow Q$. Then it would suffice to prove $\neg Q$. But if "Q" is a compound proposition, then we may need to negate it first.

Again, if you can negate the simplest compound propositions, you can negate anything.

- $\neg(P \vee Q) \equiv \neg P \wedge \neg Q$
- $\neg(P \wedge Q) \equiv \neg P \vee \neg Q$
- $\neg(P \rightarrow Q) \equiv P \wedge \neg Q$
- $\neg(\neg P) \equiv P$

Your Turn

Negate each of these compound propositions. Your final answers should have no negations of compound expressions.

- $P \rightarrow (Q \wedge R)$

- $(\neg P \vee Q) \rightarrow (S \rightarrow \neg T)$

Negating English

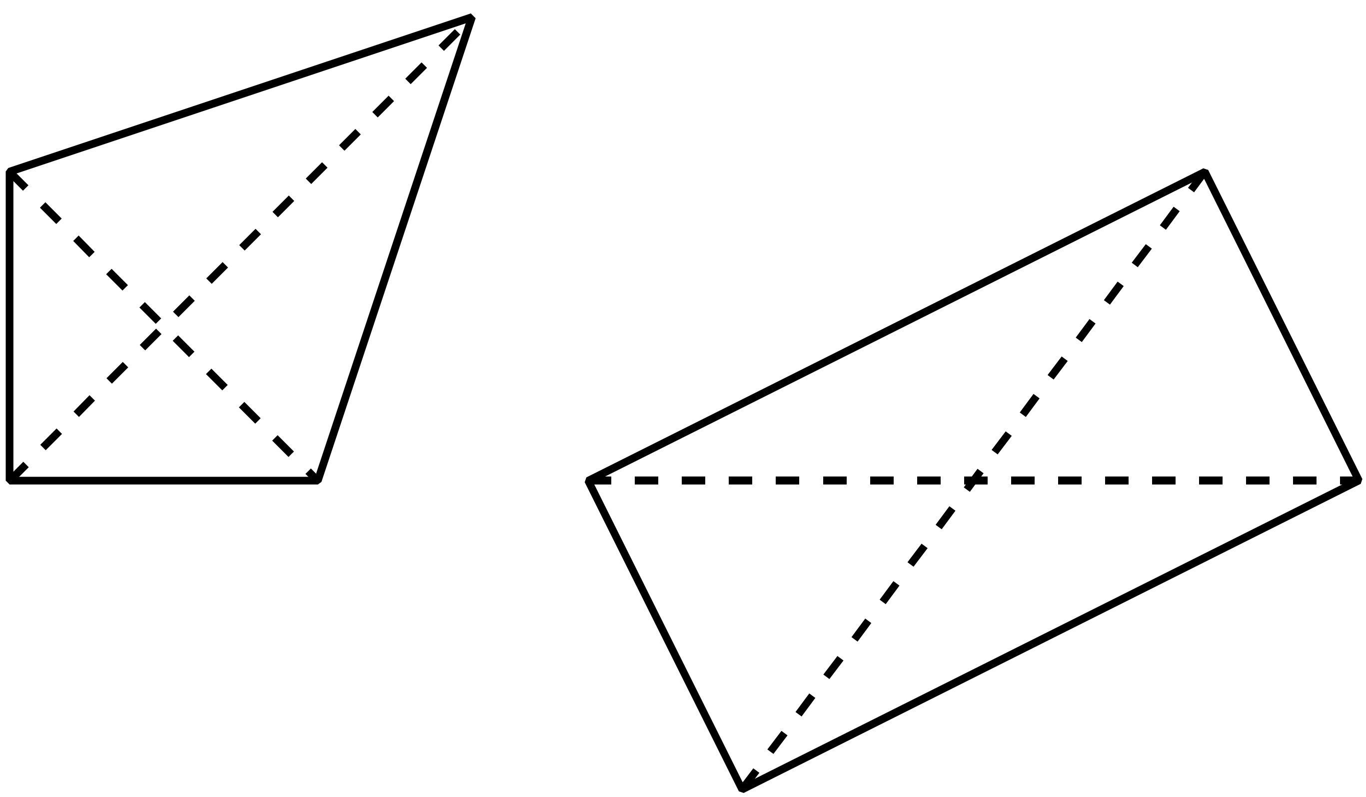
Let's try negating some English expressions.

- Quadrilateral ABCD is a trapezoid if AB is parallel to CD or BC is parallel to AD. Let us suppose ABCD is not a trapezoid. Then...

- Quadrilateral ABCD is a rhombus if its diagonals bisect one another and are perpendicular to one another. Let us suppose that ABCD is not a rhombus. Then...

Rhombus

Notice that "OR" was the right connective between "diagonals not perpendicular to one another" and "diagonals do not bisect one another."



Negating Quantified Expressions

For the third problem on the handout we needed to negate the statement:

> for every point D on *l*, the area of $\triangle$ABD equals the area of $\triangle$ABC.

This is a quantified expression of the form:

$$\forall D \; P(D)$$

What would it mean for this to be false?

$$\exists D \; \neg P(D)$$

We thus have two general rules for negating quantified expressions:

- $\neg(\forall x \; P(x)) \quad \equiv \quad \exists x \; \neg P(x)$, and
- $\neg(\exists x \; P(x)) \quad \equiv \quad \forall x \; \neg P(x)$.

Handout #1 — English from Symbols

Here are some propositions:

Symbol	Symbol
P	29 is a prime number
Q	29 is the sum of two squares
R	29 leaves a remainder of 1 when divided by 4
S	There is a perfect square which is 1 less than a multiple of 29

Please translate each of the following symbolic expressions into English.

1. $P \wedge Q$

2. $R \rightarrow S$

3. $(R \wedge P) \rightarrow Q$

4. $R \rightarrow (P \vee \neg S)$

5. $P \rightarrow [(Q \rightarrow S) \wedge (S \rightarrow Q)]$

Which of the above statements do you think are true?

Handout #2 — English from Symbols

Here are some predicates:

Symbol	Meaning	Universes of Discourse
$E(x)$	Bob eats x	x: The set of vegetables
$M(s, t)$	s has seen t	s: Americans t: The set of movies
$S(t)$	Beth has seen t	t: The set of movies
$T(a, b)$	a and b are sides of the same triangle	a, b: Sides of a triangle
$L(a, b)$	a and b have the same length	a, b: Sides of a triangle
$B(a, t)$	a belongs to t	a: Sides of a triangle t: Triangles

Please translate each of the following symbolic expressions into English.

1. E(tomatoes)	2. $\forall x\ E(x)$
3. M(Rachel, *The Princess Bride*)	4. $\forall x$ M(Rachel, x)
5. $\exists h\ \neg S(h)$	6. $\forall t\ (S(t) \rightarrow$ M(Betty, t))
7. $T(a, b) \rightarrow L(a, b)$	8. $(B(x, y) \wedge T(w, x)) \rightarrow B(w, y)$
9. $\forall a\ \forall b\ \forall c\ [(T(a, b) \wedge T(b, c)) \rightarrow T(a, c)]$	

Handout #3 — Truth Tables

Construct truth tables for each of the following compound propositions:

- $\neg Q \rightarrow \neg P$

- $(R \lor P) \land (\neg R \lor Q)$

Handout #4 — Negating Compound Propositions

Negate each of the compound propositions below. In each case, your final answer should have no negations except on the component simple propositions.

- $P \rightarrow (Q \wedge R)$

- $(\neg P \vee Q) \rightarrow (S \rightarrow \neg T)$

Handout #5 — Negating English Expressions

In these two problems you need to negate an English expression.

- Quadrilateral ABCD is a trapezoid if AB is parallel to CD or BC is parallel to AD. Let us suppose ABCD is not a trapezoid. Then...

- Quadrilateral ABCD is a rhombus if its diagonals bisect one another and are perpendicular to one another. Let us suppose that ABCD is not a rhombus. Then...

- Suppose A and B are two points in the plane and point C is not on the line *m* through A and B. Draw a line *l* through C. Then *l* is parallel to *m* if for every point D on *l*, the area of $\triangle$ABD equals the area of $\triangle$ABC. Suppose *l* is not parallel to *m*. Then ...

Exercises — Introduction to Logic

Warm-up Problems:

1. Fill in each of these simple truth tables:

P	Q	$P \wedge Q$
T	T	
T	F	
F	T	
F	F	

P	Q	$P \vee Q$
T	T	
T	F	
F	T	
F	F	

P	Q	$P \rightarrow Q$
T	T	
T	F	
F	T	
F	F	

P	$\neg P$
T	
F	

2. What should P and Q be so that "If I had a hammer, I'd hammer in the morning" is of the form $P \rightarrow Q$?

3. Negate the phrase "If I have a hammer, then I hammer in the morning."

Presentation Problems:

4. As we saw in class, if you negate an expression twice, then you end up with the expression with which you started. This sometimes has interesting consequences. For example, negate "$P \rightarrow Q$" twice using the rules we saw in class to obtain the "disjunctive form" of implication.

5. Use truth tables to show that in implication is logically equivalent to its disjunctive form, as obtained in the previous problem.

6. Build a truth table for the expression $(P \rightarrow Q) \wedge (Q \rightarrow P)$.

7. Yesterday we saw how to use syllogisms to show that some collection of premises implied some conclusion. There is a cheesier, more tedious way, too. In this method you build a truth table which includes columns for all your givens and your conclusion. Then you visually inspect to see if in all those rows in which the givens are all true, the conclusion is also true. If this is the case, then you have a (cheesy) proof of the conclusion. Use this cheesy method to prove that:
   a. P and $(P \rightarrow Q)$ imply Q.
   b. $(R \vee P)$ and $(\neg R \vee Q)$ imply $(P \vee Q)$

8. Why does the expression $(P \rightarrow Q) \rightarrow R$ need parentheses, but the expression $P \vee Q \vee R$ does not? Does the expression $P \wedge Q \wedge R$ need parentheses? How about $P \vee Q \wedge R$?

9. The resolution syllogism states that $[(R \vee P) \wedge (\neg R \vee Q)] \rightarrow (P \vee Q)$. Negate this expression, and then put into words what a counterexample to the resolution argument might look like.

10. Negate each of the following statements:
    a. Every good boy does fine
    b. All quadrilaterals have an interior angle greater than or equal to 90 degrees
    c. If the four vertices of quadrilateral ABCD lie on a circle, then angles ABC and CDA are supplementary, and angles BCD and DAB are supplementary. (*For the record, angles are supplementary if they add to 180 degrees.*)
    d. There is a triangle whose angles have integral degree measure, are in arithmetic progression, and the smallest one is 40 degrees.

Extension Problems:

11. If you haven't yet, please do the extension problems from yesterday.

12. Find a logical expression that yields the truth table shown to the right. Note that there is a *method* for doing problems like these, in which for each true row in the target column, one builds a conjunction that makes only that row true, and then takes the disjunction of these conjunctions.

P	Q	R	???
T	T	T	F
T	T	F	F
T	F	T	T
T	F	F	F
F	T	T	T
F	T	F	T
F	F	T	T
F	F	F	T

13. Define $P \Leftrightarrow Q$ to mean "$(P \rightarrow Q) \wedge (Q \rightarrow P)$." Does the expression $P \Leftrightarrow Q \Leftrightarrow R$ need parentheses?

Discrete Math Resource Book
on
Introduction to Logic

Workshop Outline — Introduction to Logic

I. Recap of symbols

A. When writing logical expressions we use various symbols for and (∧), or (∨), not (¬ or ~), implies (→) and therefore (∴)

B. Predicates are depicted in functional notation, such as P(*x*) or M(*x*, *y*).

C. Getting English from symbols is a two-step process

1. Translate the symbolic expression literally
2. Rephrase the translation so that it reads more smoothly

D. When translating predicate expressions from symbols into English, make sure to take the universe of discourse of all variables into account

II. Implications

A. The contrapositive is the converse of the inverse, and is also the inverse of the converse

B. The contrapositive is logically equivalent to the original

C. The inverse and the converse are logically equivalent to each other, since they are contrapositives of one another.

D. An implication such as P → Q can appear several ways in English:

1. "If P, then Q," "Q if P," "Whenever P, Q," "P implies Q," "P only if Q"

E. The pair of implications "P → Q" and "Q → P" can be written "P ⇔ Q" and is said "P if and only if Q" or "P iff Q." It means that they have the same truth value.

III. Compound Propositions

A. Compound propositions are built by connecting together simple propositions with ∧, ∨, →, ¬ and parentheses as needed

B. The simple propositions are the compound proposition's *component propositions*

C. The truth value of a compound proposition is determined by the truth values of its component propositions

D. We can thus build a table showing the truth value of the compound proposition under all assignments of values to its component propositions

1. This table is called the *truth table* of the compound proposition
2. It has 2*k* rows, where *k* is the number of variables present in the compound proposition
3. The truth table will frequently have "helper columns" which contain partial results in the computation of the truth values