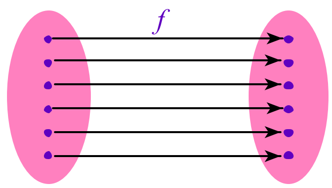
Bijections

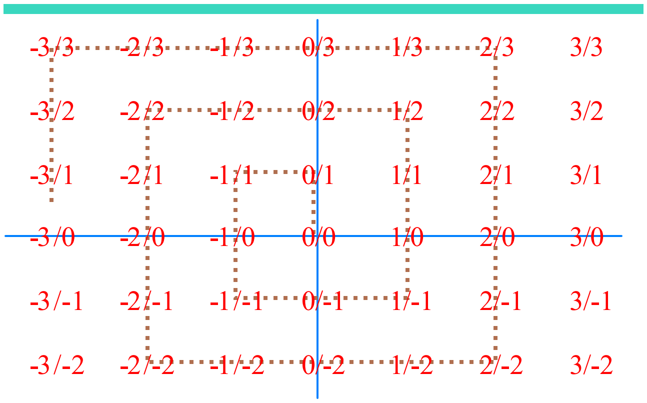
- A *bijection* is a function which is 1-1 and onto
- Two sets are said to have the same *cardinality* if there is a bijection between them
  - < Note: For finite sets this says nothing new. The cardinality of a finite set is simply the number of elements it has. But for infinite sets, it does say something new!



Bijection

- For finite sets, you cannot have a bijection between two sets unless the sets have the same cardinality (number of elements)
- For infinite sets, the story is less intuitive:
  - < Is there a bijection between the integers and the even integers: {...,!2 , !1, 0, 1, 2, ...} ø {..., !4, !2, 0, 2, 4, ...}
    - – $f(x) = 2x$
  - < ...between {0, 1, 2, 3, ...} and {1, 2, 3, ...}
    - – $f(x) = x + 1$
  - < ...between Q and ù = {0, 1, 2, 3, ...}.
    - – Q is the set of all rational numbers, that is, numbers of the form a/b, where a and b are integers, and b ..0.

Bijection between Q and ù



Bijection between Q and ù

- We discard all fractions with “0” in the denominator
- We discard all fractions not in lowest terms (to get rid of repeated fractions which are equal to each other)
- We discard all fractions with a negative denominator (to get rid of all repeats of the form 3/5 = -3/-5, or -2/7 = 2/-7)
- What’s left is a bijection between the natural numbers and the rationals, showing that they have the same cardinality

Bijection Between ù and ú

- It turns out there is no such bijection!
- Proof: Suppose there was a bijection
  - Then we would be able to create a list of *all* the real numbers that looked like this:

```
  .37489134789123819 2...     .27489134789123819 2...
19.03330102003011748 1...   19.02330102003011748 1...
23.12987341987394371 4...   23.12887341987394371 4...
  .00000001341284491 7...     .00010001341284491 7...
 4.17389417384117843 4...    4.17388417384117843 4...
-2.71834719347147193 4...   -2.71834819347147193 4...
44.37183947139438947 1...   44.37183957139438947 1...
 Etc...                      Etc...
```

This number is not on the list → .2281885...

Countable and Uncountable

- A set is called *countable* if it is
  - Finite, or
  - Has the same cardinality as the natural numbers ù
- A set is called *uncountable* if it is not countable
- Sets which are countable and infinite are called *countably infinite*
- Note that countable sets are exactly those sets that you can make a list of.
- We've seen two different sizes of infinity. It turns out that there are many more!
  - How many?

Countable or Uncountable?

- The number of finite bit strings
  - Can you make a list of all of them?
  - Yes!
    - 0
    - 1
    - 00
    - 01
    - 10
    - 11
    - 000
    - 001
    - 010
    - 001
    - 100
    - ...

Countable or Uncountable?

- The set of all infinite bit strings
  - Can you make a list of all of them?
  - No!
  - Suppose you could:
    - 01000101010101101010...
    - 10100101010101001000 01...
    - 01010010100101010010 10...
    - 11111111111110100111 1...
    - 00000000000000000000 00...
    - 11111111111111111111 11...
    - 10101011011110100000 000...
    - ...

1110100...