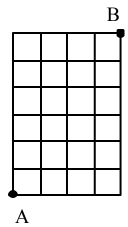
Introduction to Counting

- We will now address questions of the sort: "How many …?"
- For example:
  - How many straight flushes are possible in poker?
  - How many full houses are there?
  - How many ways are there to arrange the letters of the word "LOVE"?
  - How many ways are there to arrange the letters of the word "STRESSED"?
  - How many ways are there to walk from A to B in the diagram to the right?



The Product Rule for Counting

- To count the number of ways to perform a multi-part task, you can multiply together the number of ways to perform each part
- Question:
  - How many three-digit numbers are there?
- Answer:
  - Let us ask how many ways there are to build such a number. Building such a number consists of three parts: Selecting the first digit, selecting the second digit and selecting the third digit.
  - There are 9 ways to select the first digit, and then 10 ways to select each of the next two digits.
  - So, altogether, there are $9 \times 10 \times 10 = 900$ such numbers

Another Example

- Question:
  - How many five-letter passwords are possible if each character may be either a letter or a digit, but the first character must be a letter?
- Answer:
  - There are 26 choices for the first character
  - Each subsequent character has 36 possibilities
  - So altogether there are $26 \times 36 \times 36 \times 36 \times 36 = 43{,}670{,}016$ possible passwords

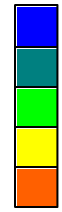
Another Example

- Question:
  - New Jersey license plates now read, left to right, three letters-two digits-one letter. Not long ago, they had three letters followed by four digits. Under which system are there more possibilities for a license plate? (Assume in each case that the initial digit cannot be 0.)
- Answer:
  - New System: $26 \times 26 \times 26 \times 9 \times 10 \times 26$
  - Old system: $26 \times 26 \times 26 \times 9 \times 10 \times 10 \times 10$
  - We don't need to compute products to determine which is bigger: We can simply take their ratio:
    - New/Old = $26 \times 26 \times 26 \times 9 \times 10 \times 26 / 26 \times 26 \times 26 \times 9 \times 10 \times 10 \times 10 = 26/100$
  - So the old system had more license plate possibilities.

Another Example

- Question:
  - How many ways are there to arrange the letters of the word "LOVE"?
- Answer:
  - We can think of arranging the letters as a process with the four parts: select the first letter, select the second letter, select the third letter, select the fourth letter.
  - How many ways to select the first letter?
    - 4
  - How many ways to select the second letter?
    - 3
  - And so on. So the total number of ways to build an arrangement of letters is $4 \cdot 3 \cdot 2 \cdot 1 = 24$.

Arranging Distinct Objects



- Question:
  - How many ways are there to stack the blocks shown to the right?
- Answer:
  - This is very similar to the preceeding problem
  - We can consider the process of building such a stack:
    - There are 5 ways to select the bottom block, 4 ways to select the block above that, etc...
  - So altogether, there are $5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$ ways to build such a stack

Arranging Distinct Objects

- It turns out that the following general question has wide application, and its answer should be memorized and understood:
  - Q: How many ways are there to arrange n distinct objects
  - A: $n \cdot (n-1) \cdot (n-2) \cdot \cdots \cdot 2 \cdot 1$
- That is, the product of the numbers from 1 to n
- This quantity is used so often, that it has its own shorthand notation: $n!$
  - Thus, $n! = n \cdot (n-1) \cdot (n-2) \cdot \cdots \cdot 2 \cdot 1$
- It is called "n factorial"

Arranging Distinct Objects

- Q: In how many orders can a quarter, dime and nickel be put into a vending machine to purchase a 40-cent soda?
- A: 3!
  - = 6
- Q: How many ways are there to line up 20 students?
- A: 20!
  - = 2432902008176640000
- Q: How many finishes were possible for the 21 horses who ran the Kentucky Derby last year?
  - 21! (assuming they all finished, and there were no ties)