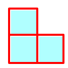
Proving Infinitely Many Theorems

- Prove that $1^2 + 2^2 + 3^2 + \cdots + n^2 = n(n + 1)(2n + 1)/6$, for all positive integers n
- Let a_0, a_1, a_2, ... be a sequence satisfying the recurrence $a_n = 5a_{n-1} - 6a_{n-2}$. Show that $a_n = 3^n - 2^n$
- Show that for all positive integers n, the $2^n \times 2^n$ checkerboard with any square deleted can be tiled with the "L"-tromino, shown to the right 
- Every set with n elements has 2^n subsets
- A post office has only 4- and 7-cent stamps. What exact amounts of postage can that post office make?

Sum of Squares

- $1^2 + 2^2 + 3^2 + \cdots + n^2 = n(n + 1)(2n + 1)/6$, for all positive integers n
  - Do you see how this is infinitely many theorems?
  - It says:
    - $1^2 = 1(1 + 1)(2\cdot1 + 1)/6$
    - $1^2 + 2^2 = 2(2 + 1)(2\cdot2 + 1)/6$
    - $1^2 + 2^2 + 3^2 = 3(3 + 1)(2\cdot3 + 1)/6$
    - ...
  - How can we prove all of those theorems
- Suppose $1^2 + 2^2 + \cdots + 80^2 = 80(81)(161)/6$ is given. How could you find the value of:
  $1^2 + 2^2 + \cdots + 80^2 + 81^2$
  without having to do all that adding?

Sum of Squares

- If we take as given that:
  - $1^2 + 2^2 + \cdots + 80^2 = 80(81)(161)/6$
- Then the task of computing
  - $1^2 + 2^2 + \cdots + 80^2 + 81^2$
- Is rather simple and straightforward.
  - $1^2 + 2^2 + \cdots + 80^2 + 81^2 =$
  - $(1^2 + 2^2 + \cdots + 80^2) + 81^2 =$
  - $80(81)(161)/6 + 81^2 =$
  - $180,441$
- We can use this same idea to prove our infinitely many theorems!

Sum of Squares

- Let us select a particular value of k
- If we take as given that the k^{th} theorem is true:
  - $1^2 + 2^2 + \cdots + k^2 = k(k + 1)(2k + 1)/6$
- The it is possible to show that the $(k + 1)^{st}$ theorem is also true:
  - $1^2 + 2^2 + \cdots + k^2 + (k + 1)^2 = (k + 1)(k + 2)(2k + 3)/6$

The thing to remember is that we are going to use this to prove that

This is what you get when you plug $k+1$ in for n in the expression: $n(n+1)(2n+1)/6$

Sum of Squares

- Show $1^2 + 2^2 + 3^2 +$ þ $+ k^2 + (k + 1)^2 =$ $(k+1)(k+2)(2k+3)/6$
- $1^2 + 2^2 + 3^2 +$ þ $+ k^2 + (k + 1)^2 =$
- $(1^2 + 2^2 + 3^2 +$ þ $+ k^2) + (k + 1)^2 =$
- $k(k + 1)(2k + 1)/6 + (k + 1)^2 =$
  - (Crack an egg to make an omelette)
- $(2k^3 + 3k^2 + k)/6 + (k^2 + 2k + 1) =$
- $(2k^3 + 3k^2 + k)/6 + (6k^2 + 12k + 6)/6 =$
- $(2k^3 + 9k^2 + 13k + 6)/6 =$
- $(k+1)(k+2)(2k+3)/6$
  - (We see that they are the same either by factoring the expression two lines up, or by multiplying out the last expression to see if it is really the same)
- Notice how most of the work is done by our assumption

Sum of Squares

- So, what about proving infinitely many theorems?
  - We have an alleged theorem for each value of n in the set {1, 2, 3, 4, 5, ...}
  - We have shown that if it is true for any particular value in that set, then it is also true for the next number in that set
  - For example, if it's true for 7, then we know it's true for 8
  - But then what else do we know?

1 2 3 4 5 6 7 8 9 10 11

- How, then, would we prove it for all positive integers n?
  - Show it is true for $n = 1$

The Whole Proof

Theorem: $1^2 + 2^2 + 3^2 +$ þ $+ n^2 = n(n + 1)(2n + 1)/6$, for all positive integers n

Proof: (By induction)

Induction Hypothesis:

Suppose the theorem is true for some particular value k:

That is, assume: $(1^2 + 2^2 + 3^2 +$ þ $+ k^2) = k(k + 1)(2k + 1)/6$

The Whole Proof

- Induction Step:
- Prove the theorem is true for the next value $k + 1$:
- That is, show: $1^2 + 2^2 + 3^2 +$ þ $+ k^2 + (k + 1)^2 = (k+1)(k+2)(2k+3)/6$
- We'll show this by considering the left-hand side:
- $1^2 + 2^2 + 3^2 +$ þ $+ k^2 + (k + 1)^2 =$
- $(1^2 + 2^2 + 3^2 +$ þ $+ k^2) + (k + 1)^2 =$
- $k(k + 1)(2k + 1)/6 \quad + (k + 1)^2 =$
  <We substituted using the induction hypothesis at this step>
- $(2k^3 + 3k^2 + k)/6 + (k^2 + 2k + 1) =$ <Multiply everything out>
- $(2k^3 + 3k^2 + k)/6 + (6k^2 + 12k + 6)/6 =$ <Common denominator>
- $(2k^3 + 9k^2 + 13k + 6)/6 =$ <Add fractions>
- $(k+1)(k+2)(2k+3)/6$
- Which is what we wanted to prove.

- Base Case:
- For the base case, we simply verify that the theorem is true when $n = 1$: Then $1^2 = 1(2)(3)/6$ is true. OK.

Recursive Sequence

- Let a_0, a_1, a_2, ... be a sequence satisfying the recurrence $a_n = 5a_{n-1} - 6a_{n-2}$, with $a_0 = 0$ and $a_1 = 1$. Show that $a_n = 3^n - 2^n$ for all $n \$ 0$.
  - Again, think of this as infinitely many theorems
  - Suppose the theorem is true for some particular value k
    - This is called the *induction hypothesis*
    - as we will discuss on the next slide, it is not enough to assume the theorem is true for k. We will need to assume more. That is, we will need to *strengthen our induction hypothesis*
  - Show that it is true for $k + 1$
    - This is called the *induction step*
  - Then prove it is true for the first theorem, when k = ???
    - 0

Recursive Sequence

- The recurrence is $a_n = 5a_{n-1} - 6a_{n-2}$.
- Show that $a_n = 3^n - 2^n$ for all $n \$ 0$.
  - Suppose it is true for k. That is: $a_k = 3^k - 2^k$
  - Show it is true for $k + 1$: That is, show $a_{k+1} = 3^{k+1} - 2^{k+1}$
  - We use the recurrence:
    - $a_{k+1} = 5\,a_k - 6\,a_{k-1} =$
    - $5(3^k - 2^k) - 6($..what..???...)
  - It is not enough to assume only the kth theorem!
  - We need to assume more.
  - The standard trick is to assume that the theorem is true not only for k, but for *all values up to and including* k.
  - (This is called the *strong* induction hypothesis)

Recursive Sequence

- The recurrence is $a_n = 5a_{n-1} - 6a_{n-2}$.
- Show that $a_n = 3^n - 2^n$ for all $n \$ 0$.
  - Suppose it is true for all values from 0 up to k. That is: $a_i = 3^i - 2^i$ for all $0 \# i \# k$
  - Show it is true for $k + 1$: That is, show $a_{k+1} = 3^{k+1} - 2^{k+1}$
  - We use the recurrence: $a_{k+1} = 5\,a_k - 6\,a_{k-1}$
  - This gives:

$$\begin{aligned} a_{k+1} &= 5(3^k - 2^k) - 6(3^{k-1} - 2^{k-1}) \\ &= (5\times3^k - 6\times3^{k-1}) - (5\times2^k - 6\times2^{k-1}) \\ &= (5\times3^k - 2\times3\times3^{k-1}) - (5\times2^k - 3\times2\times2^{k-1}) \\ &= (5\times3^k - 2\times3^k) - (5\times2^k - 3\times2^k) \\ &= 3\times3^k - 2\times2^k = 3^{k+1} - 2^{k+1} \end{aligned}$$

Which is what we wanted to prove

Recursive Sequence

- We have established that if the theorem is true for all values up to some particular value k, then it is true for the next value, $k + 1$.
- We now need to prove some base cases as a starting point for all our implications
- Let's start with $n = 0$ and $n = 1$:
  - We need to prove that $a_0 = 3^0 - 2^0$ and $a_1 = 3^1 - 2^1$. Both of these are easily verfied, as $a_0 = 0$ and $a_1 = 1$.

Tiling Squares with L-trominos

- <Please see book for a discussion of this proof by induction>