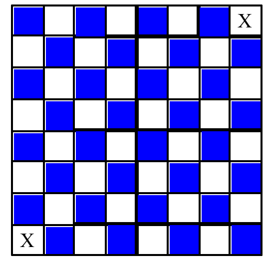
Proof by Contradiction

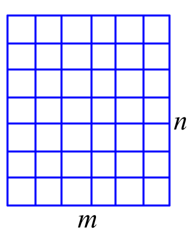
- Proof by contradiction
  - Assume:
    - $\neg p \rightarrow$ False
  - Conclude
    - p
- We show that if p is not true, then we obtain a contradiction. This implies that p must be true
- Example:
  - Prove that an 8×8 checkerboard with two opposite corners removed cannot be tiled with 1×2 dominos

Tiling a Pruned Checkerboard with Dominos



- By covering with dominos, we mean placing dominos on the checkerboard so that every square is covered without gaps or overlaps
- With these two squares removed, can what remains be covered?
  - If so, then look at what happens: Each domino, no matter where placed, covers exactly one blue and one white square. But then there must be an equal number of white and blue squares in the portion covered by dominos. This is false.

Proof by Cases



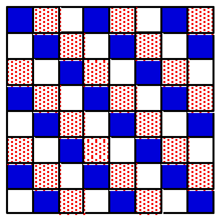
- Prove that if an $m \times n$ grid contains an even number of squares, then it can be covered by dominos
  - An $m \times n$ grid contains $m \cdot n$ squares. If this quantity is even, then either m must be even or n must be even.
    - How would you prove that?
  - If m is even...
    - Lay the dominos horizontally
  - If m is not even, then...
    - n must be even, and we can set the dominos vertically

Some Proofs

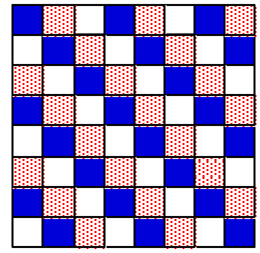
- Suppose $a > 0$ and $ab > 0$. Prove that $b > 0$.
- Proof by contradiction:
  - Suppose that $b \leq 0$. Then either $b = 0$ or $b < 0$.
    - if $b = 0$, then we would have $ab = 0$, which is false
    - if $b < 0$, then we would have $ab < 0$, which is also false
  - In either case, we get a contradiction, thus $b > 0$
- Note the use of *cases* in that proof

A Tiling Proof

- Prove that you cannot tile an 8×8 checkerboard with 1×3 trominos
  - Proof: Suppose you could. Then 64 would be a multiple of 3, which is false. (Proof by contradiction)
- Now prove that if you remove a single corner from the 8×8 checkerboard, then you cannot tile what remains with 1×3 trominos
- Hint: What if we color the checkerboard as shown to the right?
- There are 21 red, 21 white and 22 blue squares.



- We can see that if we remove a white square, then what remains cannot be tiled with 1×3 trominos
  - Because then there will be 20 white, 21 red and 22 blue squares
  - And if the region could be tiled, there would have to be the same number of each color of square
- Same for a red square
- But what if we remove a blue square? Then the resulting region *does* have an equal number of each.
- Does that mean the region *can* be tiled?
- No!



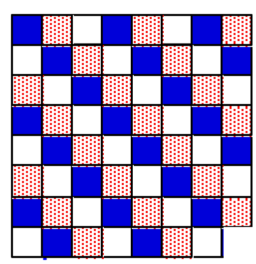
Converse versus Contrapositive

- We have the following theorem:
  - If a region of the checkerboard can be tiled with 1×3 trominos, then that a 3-coloring of that region must have the same number of each color
  - But the *converse* is not necessarily true:
    - Namely, that if a region has the same number of each color, then it can necessarily be tiled
  - In general, converses *might* be true, but you can't count on it
  - The contrapositive *is* always true, though.
    - If a 3-coloring does not have an equal number of each color, then the region cannot be tiled

---

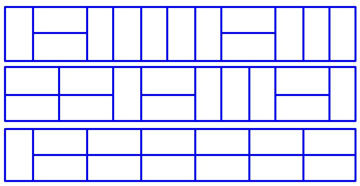
Dealing with the blue corner

- So, how do we prove the region with a blue corner removed cannot be tiled?
- Suppose it could!
  - Where do you think we go from here?
- Then we could rotate that tiling clockwise 90 degrees to obtain a tiling with the white corner removed...
- Clearly a contradiction.



Dominizing a 2×*n* rectangle

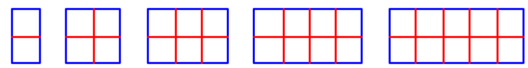
- How many different ways are there to dominize a 2×13 rectangle with 1×2 dominos?
- Here are some sample dominizations:



Can you discover a strategy for getting this answer?

A Problem-Solving Strategy

- This is not a proof-technique
- Sometimes, when you have to solve a hard problem, it makes sense to try to *solve a simpler problem*
  - How would you change the problem to make it simpler?
    - Try a different size grid...
    - Try *many* different sized grids, and see if you can find a pattern!



\# ways: 1 2 3 5 8

Proving a Conjecture

- It looks like each number in the sequence is the sum of the two previous numbers: $f(n) = f(n-1)+f(n-2)$
- How could we *prove* that this pattern will continue?
  - There are 2 ways to start dominizing a $2\times n$ grid:
    - Lay the first domino vertically
    - In which case there are $f(n-1)$ ways to finish the job
    - Lay the first domino horizontally
    - In which case the one above or below it must also lie horizontally, and then there are $f(n-2)$ ways to finish the job
  - Since those are all the cases, we've proven that $f(n) = f(n-1)+f(n-2)$

n	# ways
1	1
2	2
3	3
4	5
5	8
6	13
7	21
8	34
9	55
10	89