Chapter 1 Homework Solutions

Section 1.1

2. Propositions must be a declarative sentence with no free variables:
   a. Not a proposition – a command.
   b. Not a proposition – a question.
   c. A proposition.
   d. Not a proposition – its truth-value depends on the value of x.
   e. Not a proposition – its truth-value depends on the value of x.
   f. Not a proposition

6.
   a. If you have the flue, then you miss the final exam.
   b. You do not miss the final exam if and only if you pass the course.
   c. If you miss the final exam, then you do not pass the course.
   d. You have the flu, or miss the final exam, or pass the course.
   e. It is either the case that if you have the flu then you do not pass the course or the case that if you miss the final exam then you do not pass the course.
   f. Either you have the flu and miss the final exam, or you do not miss the final exam and do pass the course.

12.
   No. If such a barber existed, who would shave the barber? If the barber shaved himself, then he would be violating the rule that he shaves only those people who do not shave themselves. On the other hand, if he does not shave himself, then the rule says that he must shave himself. Neither is possible, so there can be no such barber.

14.
   a. If the explorer encounters a truth-teller, then he will honestly answer "no" to her question. If she encounters a liar, then the honest answer to her question is "yes", so he will lie and answer "no". Thus everybody will answer "no" to the question, and the explorer will have no way to determine which type of cannibal she is speaking to.
   b. There are several possible correct answers. One is the following question: "If I were to ask you if you always told the truth, would you say that you did?" Then if the cannibal is a truth teller, he will answer yes (truthfully), while if he is a liar, then since if fact he would have said that he did tell the truth if questioned, he will now lie and answer no.

20.
   a. Converse: If I stay home, then it will snow tonight. Contra positive: If I do not stay at home, then it will not snow tonight.
   b. Converse: Whenever I go to the beach, it is a sunny summer day. Contra positive: Whenever I do not go to the beach, it is not a sunny summer day.

c. Converse: If I sleep until noon, then I stayed up ate. Contra positive: if I do not sleep until noon, then I did not say up late.

22. (a) and (b)

P	p ⊕ q	¬p	p ⊕ ¬p
T	F	F	T
F	F	T	T

(c) and (d)

P	q	¬p	¬q	p ⊕ ¬q	¬p ⊕ ¬q
T	T	F	F	T	F
T	F	F	T	F	T
F	T	T	F	F	T
F	F	T	T	T	T

(e) and (f)

p	q	p ⊕ q	p ⊕ ¬q	(p⊕ q) ∨ (p⊕ ¬q)	(p⊕ q) ∧ (p⊕ ¬q)
T	T	F	T	T	F
T	F	T	F	T	F
F	T	T	F	T	F
F	F	F	T	T	F

26.

P	Q	R	S	p → q	(p → q) → r	((p → q) → r) → s
T	T	T	T	T	T	T
T	T	T	F	T	T	F
T	T	F	T	T	F	T
T	T	F	F	T	F	T
T	F	T	T	F	T	T
T	F	T	F	F	T	F
T	F	F	T	F	T	T
T	F	F	F	F	T	F
F	T	T	T	T	T	T
F	T	T	F	T	T	F
F	T	F	T	T	F	T
F	T	F	F	T	F	T
F	F	T	T	T	T	T
F	F	T	F	T	T	F
F	F	F	T	T	F	T
F	f	F	f	t	F	T

28.

a. Since the condition is true, the statement is executed, so x is incremented and now has the value 2.
b. Since the condition is false, the statement is not executed, so x is not incremented and now still has the value 1.
c. Since the condition is true, the statement is executed, so x is incremented and now has the value 2.
d. Since the condition is false, the statement is not executed, so x is not incremented and now still has the value 1.
e. Since the condition is true when it is encountered (since x=1) the statement is executed, so x is incremented and now has the value 2.

40. Let each letter stand for the statement that the person whose name begins with that letter is chatting. The given information can be expressed as follows:
$\neg K \to H,\ R \to \neg V, \neg R \to V,\ A \to R, V \to K,\ K \to V,\ H \to A, H \to K$.

Suppose H is true. Then it follows that A and K are true, whence it follows that R and V are true. But R implies that V is false, so we get a contradiction. Therefore h must be false. From this it follows that K is true; whence V is true, and therefore R is false, as is A. We can now check that this assignment leads to a true value for each implication. So we conclude that Kevin and Vijay are chatting but Heather, Randy, and Abby are not.

Section 1.2

4.

(a)

p	q	r	$p \vee q$	$(p \vee q) \vee r$	$q \vee r$	$p \vee (q \vee r)$
T	T	T	T	T	T	T
T	T	F	T	T	T	T
T	F	T	T	T	T	T
T	F	F	T	T	F	T
F	T	T	T	T	T	T
F	T	F	T	T	T	T
F	F	T	F	T	T	T
F	F	F	F	F	F	F

(b)

p	q	R	p ∧ q	(p ∧ q) ∧ r	q ∧ r	p ∧ (q ∧ r)
T	T	T	T	t	T	T
T	T	F	T	F	F	F
T	F	T	F	F	F	F
T	F	F	F	F	F	F
F	T	T	F	F	T	F
F	T	F	F	F	F	F
F	F	T	F	F	F	F
F	F	F	f	F	f	F

8. We construct a table for each implication and note that the relevant column contains only T's.

(a)

p	q	P ∧ q	¬ (p ∧ q)	¬ p	¬ q	¬ p ∨ ¬q
T	T	t	F	f	F	F
T	F	F	T	F	T	T
F	T	F	T	T	F	T
F	F	F	t	T	t	T

(b)

p	q	r	(p → r) -> (q → r)	q → r	[(p → q) → (q → r)] → (p → r)
T	T	T	T	T	T
T	T	F	F	T	T
T	F	T	T	T	F
T	F	F	F	F	T
F	T	T	T	T	T
F	F	T	T	T	F
F	F	F	T	T	T

(c)

p	q	p → q	p∧(p →q)	[p∧(p → q)] → q
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

(d)

p	q	r	$(p\vee q)\wedge(p\to r)\wedge(p\to r)$	$[(p\vee q)\wedge(p\to r)\wedge(p\to r)]\to r$
T	T	T	T	T
T	T	F	F	T
T	F	T	T	T
T	F	F	F	T
F	T	T	T	T
F	T	F	F	T
F	F	T	F	T
F	F	F	F	T

10. We argue directly by showing that if the hypothesis is true, then so is the conclusion. An alternative approach, which we show only for part (a), is to use the equivalences listed in the section and work symbolically.

a. Assume the hypothesis is true. Then p is false. Since $p\vee q$ is true, we conclude that q must be true. Here is a more "algebraic" solution: $[\neg p\wedge(p\vee q)]\to q \Leftrightarrow \neg[\neg p\wedge(p\vee q)]\vee q \Leftrightarrow \neg[\neg p\vee \neg(p\vee q)]\vee q \Leftrightarrow$ $p\vee\neg(p\vee q)\vee q \Leftrightarrow (p\vee q)\vee\neg(p\vee q) \Leftrightarrow \mathbf{T}$. The reasons for these logical equivalences are, respectively, Table 6, line 3; De Morgan's law; double negation; commutative and associative laws; Table 6, line 1.

b. We want to show that if the entire hypothesis is true, then the conclusion $p\to r$ is true. To do this, we need only show that if p is true, then r is true. Suppose p is true. Then by the first part of the hypothesis, we conclude that q is true. It now follows from the second part of the hypothesis that r is true, as desired.

c. Assume the hypothesis is true. Then p is true, and since the second part of the hypothesis is true, we conclude that q is also true, as desired.

d. Assume the hypothesis is true. Since the first part of the hypothesis is true, we know that either p or q is true. If p is true, then the second part of the hypothesis tells us that r is true; similarly, if q is true, then the third part of the hypothesis tells us that r is true. Thus in either case we conclude that r is true.

20. We apply the rules stated in the preamble.

a. $p\vee\neg q\vee\neg r$

b. $(p\vee q\vee r)\wedge s$

c. $(p\wedge\mathbf{T})\vee(q\wedge\mathbf{F})$

26. The statement of the problem is really the solution. Each line of the truth table corresponds to exactly one combination of truth-values for the n atomic propositions involved. We can write down a conjunction that is true precisely in this case, namely the conjunction of all the atomic propositions that are true and the negations of all the atomic propositions that are false. If we do this for each line of the truth table for which the value of the compound proposition is to be true, and take the disjunction of the resulting propositions, then we have the desired proposition in its disjunctive normal form.

Section 1.3

2.

a. This is true, since there is an a in orange.
b. This is false, since there is no a in lemon.
c. This is false, since there is no a in true.
d. This is true, since there is an a in false.

6.

a. Some student in your class has taken some computer science courses.
b. There is a student in your class who has taken every computer science course.
c. Every student in your class has taken at least one computer science course.
d. There is a computer science course that every student in your class has taken
e. Every computer science course has been taken by at least one student in your class.
f. Every student in your class has taken every computer science course.

8.

a. Randy Goldberg is enrolled in CS 252.
b. Someone is enrolled in Math 695.
c. Carol Sitea is enrolled in some course.
d. Some student is enrolled simultaneously in Math 222 and CS 252.
e. There exist two distinct people, the second of whom is enrolled in every course that the first is enrolled in.
f. There exist two distinct people enrolled in exactly the same courses.

20.

a. This is false, since $1 + 1 \neq 1 - 1$.
b. This is true, since $2 + 0 = 2 - 0$.
c. This is false, since there are many values of y for which $1 + y \neq 1 - y$.
d. This is false, since the equation $x + 2 = x - 2$ has no solution.
e. This is true, since we can take $x = y = 0$.
f. This is true, since we can take $y = 0$ for each x.
g. This is true, since we can take $y = 0$.
h. This is false, since part (d) was false.
i. This is certainly false.

40. It is enough to find a counterexample. It is intuitively clear that the first proposition is asserting much more than the second. It is saying that one of the two predicates, P or Q, is universally true; whereas the second proposition is simply saying that for every x either P(x) or Q(x) holds, but which it is may well depend on x. As a simple counterexample, let P(x) be the statement that x is odd,

and let Q(x) be the statement that x is even. Let the universe of discourse be the positive integers. The second proposition is true, since every positive integer is either odd or even. But the first proposition is false, since it is neither the case that all positive integers are odd nor the case that all of them are even.

Section 1.4

2. There are of course an infinite number of correct answers.
   a. $\{3n \mid n = 0, 1, 2, 3, 4\}$ or $\{x \mid x \text{ is a multiple of } 3 \wedge 0 \leq x \leq 12\}$.
   b. $\{x \mid -3 \leq x \leq 3\}$, where we are assuming that the universe of discourse is the set of integers.
   c. {x | x is a letter of the word monopoly other than l or y}.

4. Each of the sets is a subset of itself. Aside form that, the relations are $B \subseteq A$ and $C \subseteq D$ and $C \subseteq A$.

6.
   a. Since the set contains only integers and {2} is a set, not an integer, {2} is not an element.
   b. Since the set contains only integers and {2} is a set, not an integer, {2} is not an element.
   c. The set has two elements. One of them is patently {2}.
   d. The set has two elements. One of them is patently {2}.
   e. The set has two elements. One of them is patently {2}.
   f. The set has only one element, {{2}}; since this is not the same as {2} (the former is a set containing a set, whereas the latter is a set containing a number), {2} is not an element of {{2}}.

12. The cardinality of a set is the number of elements it has.
   a. The empty set has no elements, so its cardinality is 0.
   b. This set has one element (the empty set), so its cardinality is 1.
   c. This set has two elements, so its cardinality is 2.
   d. This set has three elements, so its cardinality is 3.

14. The union of all the sets in the power set of a set X must be exactly X. In other words, we can recover X from its power set, uniquely. Therefore the answer is yes.

Section 1.5

2. a). $A \cap B$
   b). $A \cap$ ‘B
   c). $A \cup B$
   d). ‘A $\cup$ ‘B

4. Note that $A \subseteq B$.
   a. {a, b, c, d, e, f, g, h} = B
   b. {a, b, c, d, e} = A
   c. There are no elements in A that are not in b, so the answer is $\varnothing$

10. a. If x is in $A \cap B$, the perforce it is in A { by definition of intersection).
   b. If x is in A, then perforce it is in $A \cup B$ (by definition of union).
   c. If x is in $A - B$, then perforce it is in A (by definition of difference).
   d. If $x \in A$ then $x \notin B - A$. Therefore there can be no elements in $A \cap (B - A)$, so $A \cap (B - A) = \varnothing$.
   e. The left-hand side consist precisely of those things that are either elements of A or else elements of B but no t A, in other words, things that are elements of either A or B (or, of course, both). This is precisely the definition of the right-hand side.

14. That $A \subseteq (A \cap B) \cup (A \cap$ 'B) follows from the fact that every element $x \in A$ is an element of either $A \cap B$ (if $x \in B$) or $A \cap$ 'B (if $x \notin B$). On the other hand, if $x \in (A \cap B) \cup (A \cap$ 'B), then either $x \in A \cap B$ or $x \in A \cap$ 'B. in either case, $x \in A$ by the definition of intersection.

18.

22. This is the set of elements in exactly one of these sets, namely {2, 5}.

28. a. This is clear form the symmetry (between A and B) in the definition of symmetric difference.

   b. We prove two things. To show that $A \subseteq (A \oplus B) \oplus B$, suppose $x \in A$. If $x \in B$, then $x \notin A \oplus B$, so x is an element of the right-hand side. On the other hand if $x \notin B$, then $x \in A \oplus B$, so again x is in the right-hand side. Conversely, suppose x is an element of the right-hand side. There are two cases. If $x \notin B$, then necessarily $x \in A \oplus B$, whence $x \in A$. If $x \in B$, then necessarily $x \notin A \oplus B$, and the only way for that to happen (since $x \in B$) is for x to be in A.

34. To count the elements of $A \cup B \cup C$ we proceed as follows. First we count the elements in each of the sets and add. This certainly gives us all the elements in the union, but we have over counted. Each element in $A \cap B$, $B \cap C$, and $A \cap C$ has been counted twice. Therefore we subtract the cardinalities of these intersections to make up for the over count. Finally, we have compensated a bit too much, since the elements of $A \cap B \cap C$ have now been counted three times and subtracted three times. We adjust by adding back the cardinality of $A \cap B \cap C$.

Section 1.6

2. a. This is not a function because the rule is not well defined. We no not know whether f (3) = 3 or f (3) = -3.
   b. This is a function. For all integers n, the square root of $n^2 + 1$ is a well-defined real number.
   c. This is not a function with domain Z, since for n = 2 (and also for n = -2) the value of f (n) is not defined by the given rule. In other words, f (2) and f (-2) are not specified since division by 0 makes no sense.

4. a. The domain is the set of nonnegative inters, and the range is the set of digits (0 through 9).
   b. The domain is the set of positive integers, and the range is the set of integers greater than 1.
   c. The domain is the set of all bit strings, and the range is the set of nonnegative integers.
   d. The domain is the set of all bit strings, and the range is the set of nonnegative integers (a bit string can have length 0).

8. a. This is one-to-one
   b. This in not one-to-one, since b is the image of both a and b.
   c. This is not one-to-one, since d is the image of both a and d.

Section 1.7

2. Just plug "8" into the expressions given
   a. 28-1 = 128
   b. 7 (Note, there is nothing to plug in here.)
   c. 2
   d. -256

4. a. 1, -2, 4, -8j
   b. 3, 3, 3, 3
   c. 8, 11, 23, 71
   d. 2, 0, 4, 0

6. a. 10, 7, 4, 1, -2, -5, -8, -11, -14, -17
   b. 1, 3, 6, 10, 15, 21, 28, 36, 45, 55 (assuming we start with 1)
   c. 1, 5, 19, 65, 211, … (assuming we start with 1)
   e. 1, 2, 3, 5, 8, 13, 21, 34, 55, 89

10 a. Start with 3 and each time add the next odd number, starting with 3. Or $a_n = n^2 + 2$, again assuming the sequence starts with a_1

b. Start with 7 and add 4 each time. Or $a_n = 4n + 3$

f. Start with 1 and multiply by the successive odd numbers each time, beginning with 3. (We can't write a good formula for this at this time, but eventually we'll write $(2n\text{-}1)!/2^{n\text{-}1}(n\text{-}1)!$

34. If the subset is finite, then it is countable. If the subset is infinite then we know the set itself is countably infinite. The idea is to make a list of all the elements of the countable set, and then to go through the list and map each element of the list that we come across which is also in B to successive natural numbers, starting with 0. This gives the desired bijection.

32. In the first 2 cases we have subsets of the integers, and the integers are a countable set (as can be seen by listing them as follows: 0, 1, -1, 2, -2, 3, -3, 4, -4, …) and so they are countable. Bijections can be made by simply going through the list shown in the line above the way we did problem 34. Part c. has a more interesting bijection, where we list first the number .1111111…, then all numbers with one "1" in their decimal expansion (such as 1 and .1), then the number 1.1111111…, then all numbers with two "1"s in their decimal expansion (11, 1.1, .11), then 11.1111…, etc…

For part d. we have an uncountable set. You can see this by taking only those numbers between 0 and 1 (non-inclusive) whose decimal expansions are infinite strings of 1s and 9s, and then using Cantor's diagonal argument.