The Choose Numbers

- How many ways are there to select k objects from n distinct objects?
  - We called that number "n choose k"
  - We denoted it $\binom{n}{k}$
- I'll informally call such numbers the "choose numbers"

Some Sample Choose Numbers

- What is 4 choose 1?
  - 4
  - What is 4 choose 2?
    - 6
  - What is 5 choose 1?
    - 5
  - What is 5 choose 2?
    - 10
  - What is 5 choose 3?
    - 10
    - Note that this is the same as 5 choose 2.
    - That's because selecting which three to pick is the same as selecting which two not to pick
    - And thus there are the same numbers of ways to make the selections

Some Sample Choose Numbers

- What is 5 choose 4
  - Same as 5 choose 1, which is 5
- In general, $\binom{n}{k} = \binom{n}{n-k}$
- In general, what is n choose 1?
  - n
- What is n choose n?
  - 1
- What is n choose 0?
  - 1
- What is n choose 2?
  - $n(n - 1) / 2$, which can be derived from the factorial expression for choose numbers

Choose Polynomials

- For fixed k, the value of n choose k is a polynomial in n of degree k
- For example:

$$\binom{n}{0} = 1$$

$$\binom{n}{1} = n$$

$$\binom{n}{2} = (1/2)n^2 - (1/2)n$$

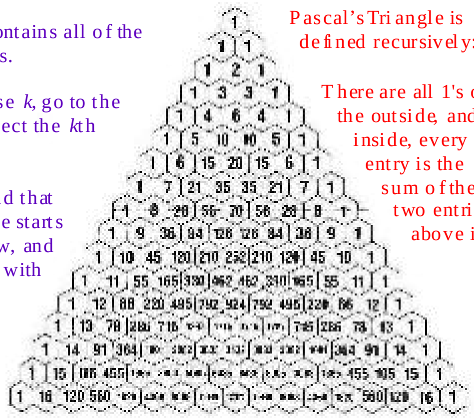
$$\binom{n}{3} = (1/6)n^3 - (1/2)n^2 + (1/3)n$$

Pascal’s Triangle

This triangle contains all of the choose numbers.

To find *n* choose *k*, go to the *n*th row and select the *k*th entry.

But bear in mind that Pascal’s triangle starts with the 0th row, and each row starts with the 0th entry.



Pascal’s Triangle is defined recursively:

There are all 1's on the outside, and inside, every entry is the sum of the two entries above it.

Binomial Coefficients

- The rest of the world calls the choose numbers “binomial coefficients”
- That is because they appear in the expansion of binomials to integer powers
- 1
- $a + b$
- $a^2 + 2ab + b^2$
- ...

Binomial Coefficients are Choose Numbers

$$(a + b)^5$$
$$=$$
$$(a + b)\cdot(a + b)\cdot(a + b)\cdot(a + b)\cdot(a + b)$$

In the first expression, the coefficient of a^2b^3, for example, is what we call a binomial coefficient.

To compute the coefficient of a^2b^3 in the second expression, we count the number of ways to select a term from each of the five factors, making sure that exactly 3 of them are “ *b*”. There are 5 choose 3 ways to do that.

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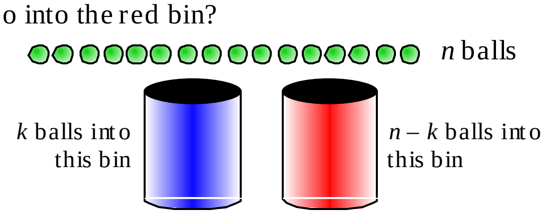
Another Question

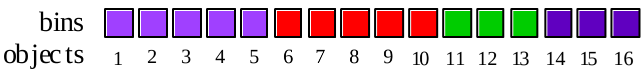
- How many ways are there to place *n* balls into two bins, one bin blue and the other bin red, so that exactly *k* balls go into the blue bin and $n - k$ go into the red bin?



< The answer is *n* choose *k*, since we can simply count the number of ways to select the *k* for the blue bin.

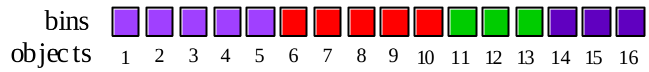
More Bins

- How many ways are there to place n distinct objects into t distinct bins so that
  - the number of objects in each bin is $k_1, k_2, \ldots, k_t$
  - where $k_1 + k_2 + \cdots + k_t = n$?
- Can you think of a "choose number" way to do this?
- Here is an anagram way to do this:



- Every rearrangement of the bins gives another assignment of objects to bins.

Multinomial Coefficients



- Every rearrangement of the bins gives another assignment of objects to bins.
- The number of "bin anagrams" is 16! / 5!5!3!3!
- In general, there are $n! / k_1!k_2!\ldots k_t!$
- These number are called multinomial coefficients

Some Sample Problems

- How many ways are there to divide twenty students into four teams, the A, B, C and D teams, each containing five students?
  - 20! / 5!5!5!5! = 11,732,745,024
- Six students were elected as officers, and the teacher wishes to assign one of those students as president, two as vice presidents and three as secretaries. How many was are there to do this?
  - 6!/1!2!3! = 60