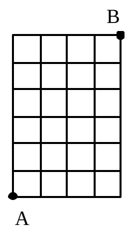
Introduction to Counting

- We will now address questions of the sort: "How many ...?"
- For example:
  - How many straight flushes are possible in poker?
  - How many full houses are there?
  - How many ways are there to arrange the letters of the word "LOVE"?
  - How many ways are there to arrange the letters of the word "STRESSED"?
  - How many ways are there to walk from A to B in the diagram to the right?



The Product Rule for Counting

- To count the number of ways to perform a multi-part task, you can multiply together the number of ways to perform each part
- Question:
  - How many three-digit numbers are there?
- Answer:
  - Let us ask how many ways there are to build such a number. Building such a number consists of three parts: Selecting the first digit, selecting the second digit and selecting the third digit.
  - There are 9 ways to select the first digit, and then 10 ways to select each of the next two digits.
  - So, altogether, there are $9 \times 10 \times 10 = 900$ such numbers

Another Example

- Question:
  - How many five-letter passwords are possible if each character may be either a letter or a digit, but the first character must be a letter?
- Answer:
  - There are 26 choices for the first character
  - Each subsequent character has 36 possibilities
  - So altogether there are $26 \times 36 \times 36 \times 36 \times 36 = 43{,}670{,}016$ possible passwords

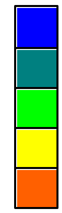
Another Example

- Question:
  - New Jersey license plates now read, left to right, three letters-two digits-one letter. Not long ago, they had three letters followed by four digits. Under which system are there more possibilities for a license plate? (Assume in each case that the initial digit cannot be 0.)
- Answer:
  - New System: $26 \times 26 \times 26 \times 9 \times 10 \times 26$
  - Old system: $26 \times 26 \times 26 \times 9 \times 10 \times 10 \times 10$
  - We don't need to compute products to determine which is bigger: We can simply take their ratio:
    - New/Old = $26 \times 26 \times 26 \times 9 \times 10 \times 26 / 26 \times 26 \times 26 \times 9 \times 10 \times 10 \times 10 = 26/100$
  - So the old system had more license plate possibilities.

Another Example

- Question:
  - How many ways are there to arrange the letters of the word "LOVE"?
- Answer:
  - We can think of arranging the letters as a process with the four parts: select the first letter, select the second letter, select the third letter, select the fourth letter.
  - How many ways to select the first letter?
    - 4
  - How many ways to select the second letter?
    - 3
  - And so on. So the total number of ways to build an arrangement of letters is $4 \cdot 3 \cdot 2 \cdot 1 = 24$.

Arranging Distinct Objects



- Question:
  - How many ways are there to stack the blocks shown to the right?
- Answer:
  - This is very similar to the preceeding problem
  - We can consider the process of building such a stack:
    - There are 5 ways to select the bottom block, 4 ways to select the block above that, etc...
  - So altogether, there are $5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$ ways to build such a stack

Arranging Distinct Objects

- It turns out that the following general question has wide application, and its answer should be memorized and understood:
  - Q: How many ways are there to arrange n distinct objects
  - A: $n \cdot (n-1) \cdot (n-2) \cdot \cdots \cdot 2 \cdot 1$
- That is, the product of the numbers from 1 to n
- This quantity is used so often, that it has its own shorthand notation: $n!$
  - Thus, $n! = n \cdot (n-1) \cdot (n-2) \cdot \cdots \cdot 2 \cdot 1$
- It is called "n factorial"

Arranging Distinct Objects

- Q: In how many orders can a quarter, dime and nickel be put into a vending machine to purchase a 40-cent soda?
- A: 3!
  - = 6
- Q: How many ways are there to line up 20 students?
- A: 20!
  - = 2432902008176640000
- Q: How many finishes were possible for the 21 horses who ran the Kentucky Derby this year?
  - 21! (assuming they all finished, and there were no ties)

Arranging Objects with Repeats

- How many ways are there to arrange the letters of the word "LONE"?
  - 4! = 24, because the letters were all distinct
- How many ways are there to arrange the letters of the word "NONE"?
  - The answer is not 4! anymore.
  - How many ways to select the first letter?
    - 3
  - How many ways to select the second letter?
    - The answer depends on our selection for the first letter!
  - So, not only is the answer not 4!, but we can't even use a straightforward multiplication for this problem at all.

Arranging Objects with Repeats

- The trick is to *color* the letters, to make them all different...
  - "NONE" has some repeats
  - But "NONE" has none

There are 24 arrangements of the word "NONE", because now the letters are distinct.

But if we remove the colors from the letters, those 24 arrangements are not all distinct anymore.

"NONE" = "NONE", and "ONNE" = "ONNE"

- How many distinct arrangements will there be?