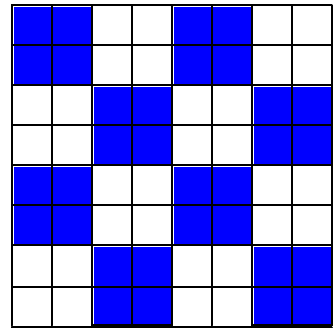
Methods of Proof

- Direct Proof
  - Assume:
    - p
    - p 6 q
  - Conclude
    - q
- In proofs of this type, the bulk of the work is in showing p 6 q
- For example:
  - Prove that if I put 17 kings onto a 8×8 chessboard, then there must be two kings that are adjacent, either horizontally, vertically or diagonally.

Kings on a Chessboard

- Suppose I 2-color the chessboard as shown
- There are 16 2×2 regions
- If I place 17 kings into these 16 regions, then some region will get at least 2 kings
- They will be adjacent
- End of proof. •



Indirect Proof

- Indirect Proof
  - Assume
    - ¬q 6 ¬p
  - Conclude
    - p 6 q
- This is simply the assertion that an implication is logically equivalent to its contrapositive
- For example:
  - Prove that if *n* is congruent to 3 (mod 4), then *n* is not the sum of two squares
    - p: *n* is congruent to 3 (mod 4)
    - q: *n* is not the sum of two squares

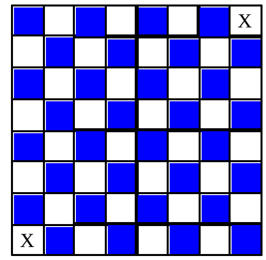
Sum of Two Squares

- Prove that if *n* is congruent to 3 (mod 4), then *n* is not the sum of two squares
  - p: *n* is congruent to 3 (mod 4)
  - q: *n* is not the sum of two squares
- We will show that ¬q 6 ¬p
  - Assume ¬q. Then *n* is the sum of two squares:
    - $n = a^2 + b^2$, where *a* and *b* are integers
  - Let's consider the possible values of a square (mod 4):
    - Every integer is either 0, 1, 2 or 3 (mod 4)
    - The squares of those are 0, 1, 4, 9 (mod 4), which reduce to:
    - 0, 1, 0, 1 (mod 4)
    - That is, every square is congruent to either 0 or 1 (mod 4)
  - But then the sum $a^2 + b^2$ is either 0, 1 or 2
  - But *not* 3. So *n* is *not* congruent to 3 (mod 4)

Proof by Contradiction

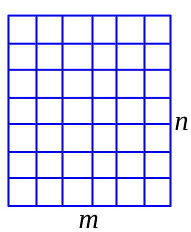
- Proof by contradiction
  - Assume:
    - $\neg p \Rightarrow$ **False**
  - Conclude
    - p
- We show that if p is not true, then we obtain a contradiction. This implies that p must be true
- Example:
  - Prove that an 8×8 checkerboard with two opposite corners removed cannot be tiled with 1×2 dominos

Tiling a Pruned Checkerboard with Dominos



- By covering with dominos, we mean placing dominos on the checkerboard so that every square is covered without gaps or overlaps
- With these two squares removed, can what remains be covered?
  - If so, then look at what happens: Each domino, no matter where placed, covers exactly one blue and one white square. But then there must be an equal number of white and blue squares in the portion covered by dominos. This is false.

Proof by Cases



- Prove that if an $m \times n$ grid contains an even number of squares, then it can be covered by dominos
  - An $m \times n$ grid contains $m \cdot n$ squares. If this quantity is even, then either m must be even or n must be even.
    - How would you prove that?
  - If m is even...
    - Lay the dominos horizontally
  - If m is not even, then...
    - n must be even, and we can set the dominos vertically