

Applications of Orthogonal Arrays to Computer Science

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East Carolina University

AND

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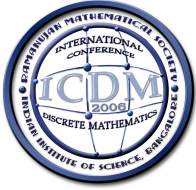
Greenville, NC, USA

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ICDM'06

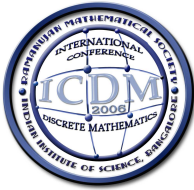
Indian Institute of Science

17th Dec. 2006



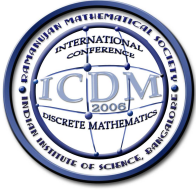
Orthogonal Arrays

- An *orthogonal array* $OA_{\lambda}(t, k, v)$ is a $\lambda v^t \times k$ array of symbols from a v -set, such that in any t columns, every possible list of t symbols occurs in exactly λ rows.



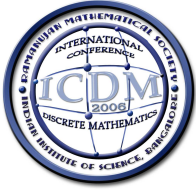
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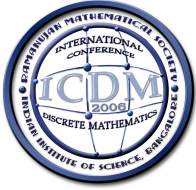
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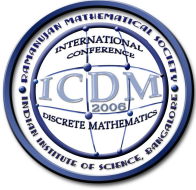
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- If $v = 2$, then these are called *binary orthogonal arrays*.



An Example

Here is a simple $OA(3, 4, 2)$.

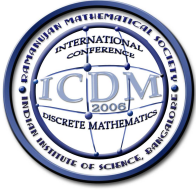
1	2	3	4
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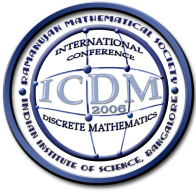
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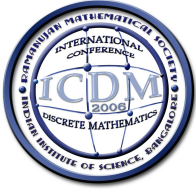
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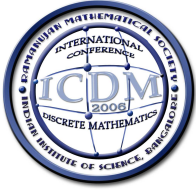
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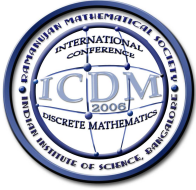
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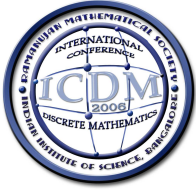
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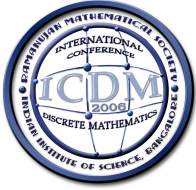
Theory of OAs

- For what parameters t , k , v and λ , do they exist?



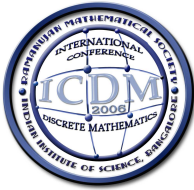
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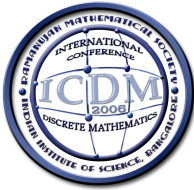
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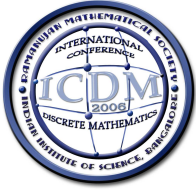
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- For basic theory of OAs, see the recent book on Orthogonal Arrays by Hedayat, Sloane and Stuffken.



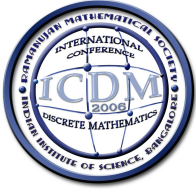
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- Here we will focus on Applications of OAs to Computer Science.



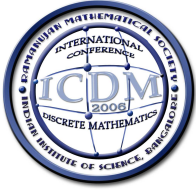
Applications of OAs

- Threshold Schemes



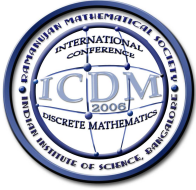
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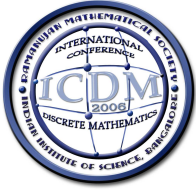
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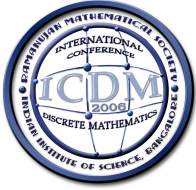
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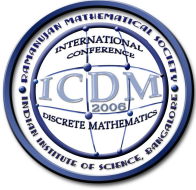
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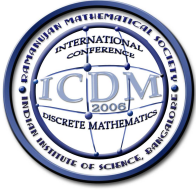
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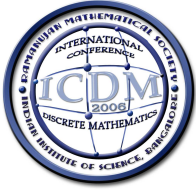
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- and many more.



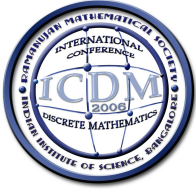
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- A method of sharing a secret among a set of n participants so that only groups of participants of size at least t could gain access to the secret.



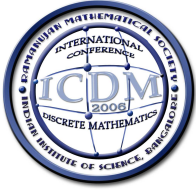
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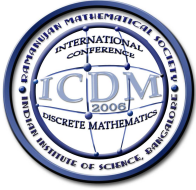
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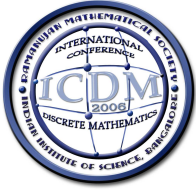
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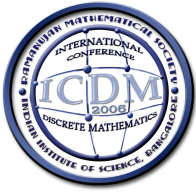
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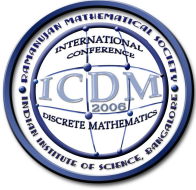
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- The shares should be distributed secretly. Let \mathcal{S} be the set of possible values of the shares.



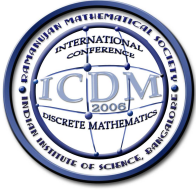
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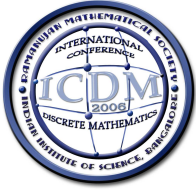
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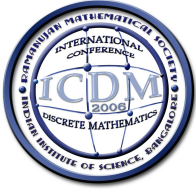
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 2. On the other hand, if $|B| < t$, then they should be able to determine nothing about the value of K .



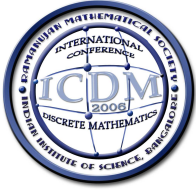
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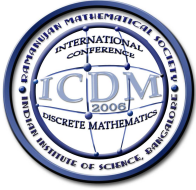
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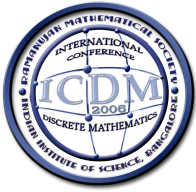
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- A perfect threshold scheme in which $|\mathcal{S}| = |\mathcal{K}|$, is called an *ideal threshold schemes*.



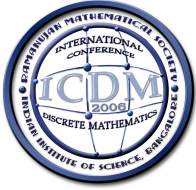
Relation to OAs

- **Theorem:** An ideal (t, n) threshold scheme with $|\mathcal{K}| = v$ exists if and only if an $OA(t, n + 1, v)$ exists.



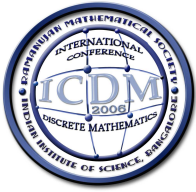
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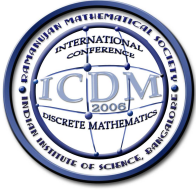
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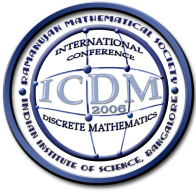
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 - Fairly simple; we shall prove half of it.



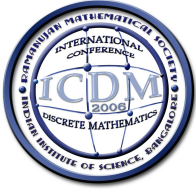
Construction

- Start with $OA(t, n + 1, v)$.



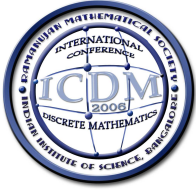
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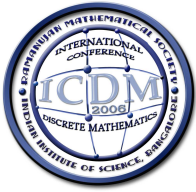
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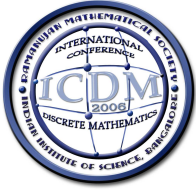
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- To distribute a specific key K , dealer selects a random row of OA such that K appears in the first column.



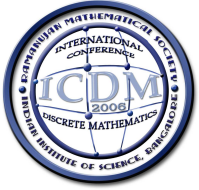
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- Dealer gives out remaining elements of the row as shares to the participants.



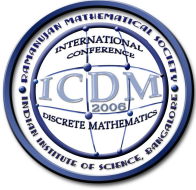
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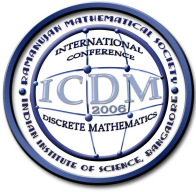
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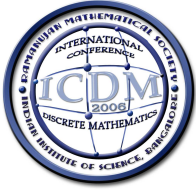
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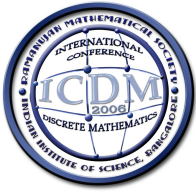
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- Can a group of $t - 1$ participants compute K ?
- Any possible value of secret along with shares of $t - 1$ participants determine a unique row of the OA.
- Hence, $t - 1$ participants can get no information about the secret.



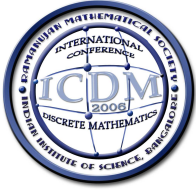
Authentication Codes - Idea

- Alice wants to communicate to Bob over a public channel.



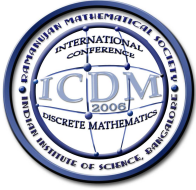
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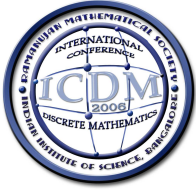
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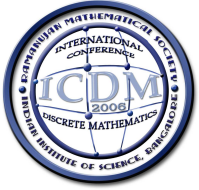
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- When Bob receives a message from Alice, How can he be sure that the message was really sent by Alice and is not tampered with along the way?



Authentication Codes - Definition

- An **Authentication Code** is a four-tuple $(S, A, \mathcal{K}, \mathcal{E})$, where



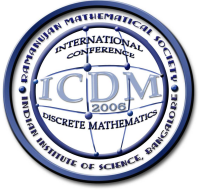
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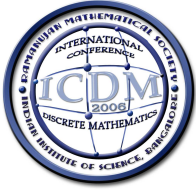
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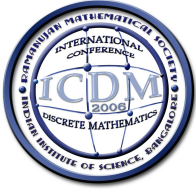
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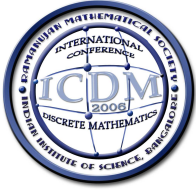
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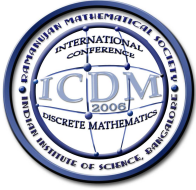
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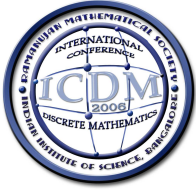
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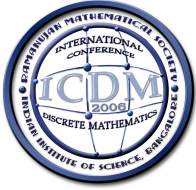
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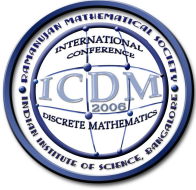
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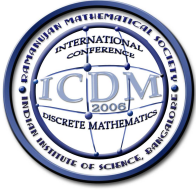
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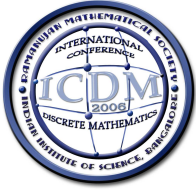
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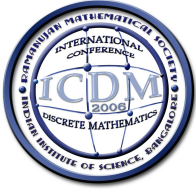
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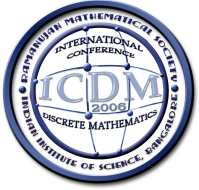
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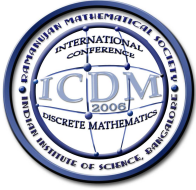
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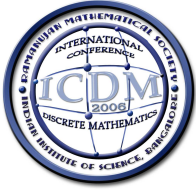
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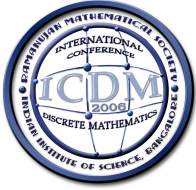
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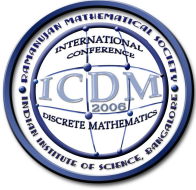
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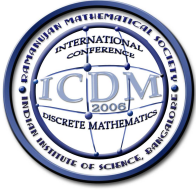
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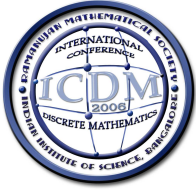
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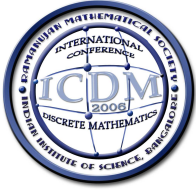
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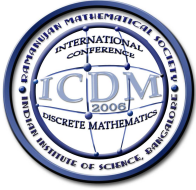
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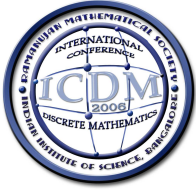
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 - Also, minimize the number of authentication rules.
- It is not too difficult to show that $P_{d_0} \geq 1/l$ and $P_{d_1} \geq 1/l$, where l is the number of authenticators.



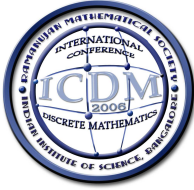
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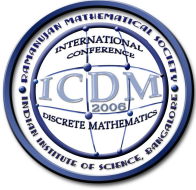
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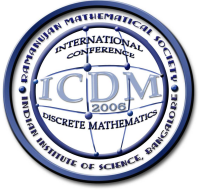
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 2. $|\mathcal{E}| \geq k(l - 1) + 1$, and equality occurs if and only if the authentication matrix is an $OA_\lambda(2, k, l)$ where

$$\lambda = \frac{k(l - 1) + 1}{l^2}.$$



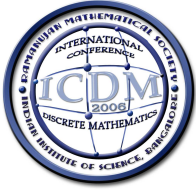
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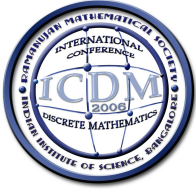
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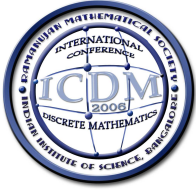
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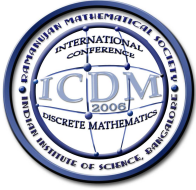
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Illustration

4 states, 3 authenticators, 9 encoding rules.

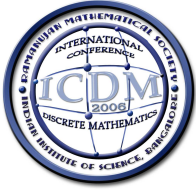
	s_1	s_2	s_3	s_4
1	1	1	1	1
1	1	2	2	2
1	1	3	3	3
2	2	1	2	3
2	2	2	3	1
2	2	3	1	2
3	3	1	3	2
3	3	2	1	3
3	3	3	2	1



Illustration

Suppose $(s_2, 3)$ is observed by Oscar.

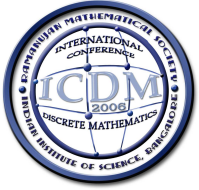
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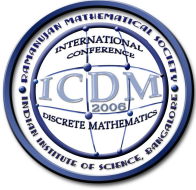
Suppose Oscar wants to substitute s_4 .

	s_1	s_2	s_3	s_4
1	1	1	1	1
1	1	2	2	2
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2	2	1	2	3
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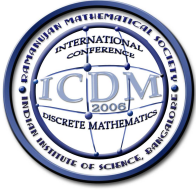
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 - Monte Carlo algorithm*** always gives an answer, but the answer may be incorrect with some probability ϵ .



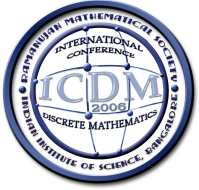
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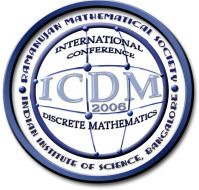
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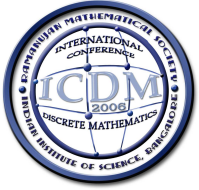
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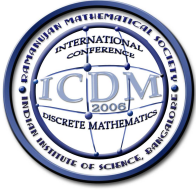
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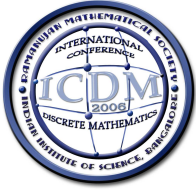
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- 0 means *No* and 1 means *Yes*.



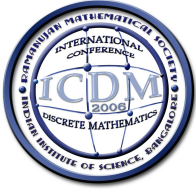
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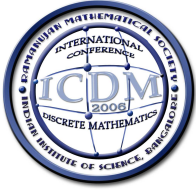
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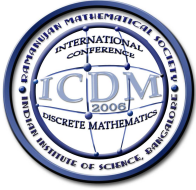
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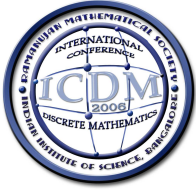
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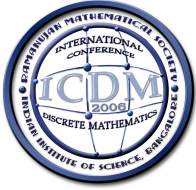
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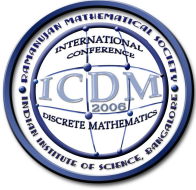
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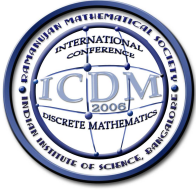
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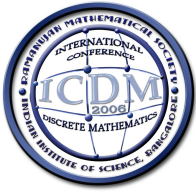
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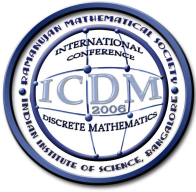
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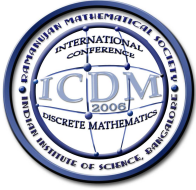
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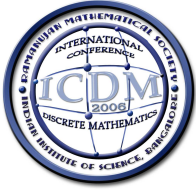
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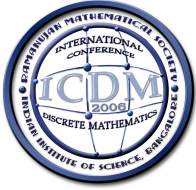
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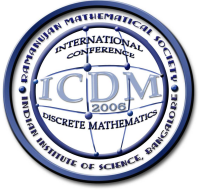
Use of OAs

- Here is an alternative approach.
- Consider an $OA(2, k, n)$.
- This array has n^2 rows.
- Generate $2 \log n$ true random bits.
- Use them to index a specific row of the OA.
- Run the Monte Carlo Algorithm (k times) using the k elements in the row selected as the sample points.



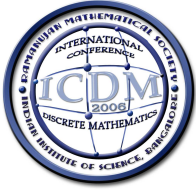
Calculation of Error Probability

- Let U denote the universe of sample points;
 $|U| = n$.



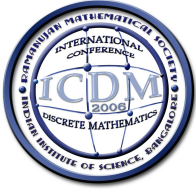
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- Let U denote the universe of sample points;
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- Let $S \subseteq U$ be the set of *witnesses*; Then
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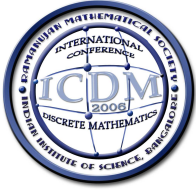
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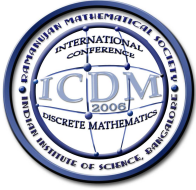
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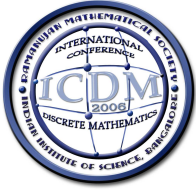
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 $|S| \geq (1 - \epsilon)n$.
- Call a row of the OA a *bad row* if none of the elements in the row is a witness.
- Then error probability is simply the probability that a randomly selected row of the OA is bad.
- Can be shown to be at most $\frac{\epsilon}{1 + (k-1)(1-\epsilon)}$ using combinatorial properties of OAs.



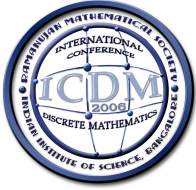
Remarks

- A similar idea was first proposed by Chor and Goldreich under the name *two-point based sampling*. They used a *specific* OA and derived bounds on error probability by using complex techniques.



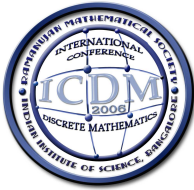
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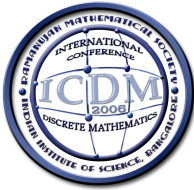
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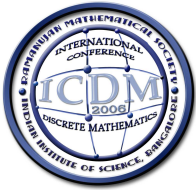
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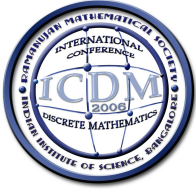
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- The scheme presented is a generalization that
 - works for any OA.
 - yields better error probability.
 - is analyzable by elementary techniques.



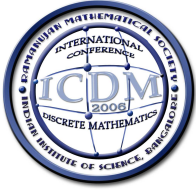
A comparison

Name	# ran. bits	Error Prob.
Original Scheme	$k \log n$	ϵ^k
Two-Point Scheme	$2 \log n$	$\frac{\epsilon}{(1-\epsilon)k}$
OA Scheme	$2 \log n$	$\frac{\epsilon}{1+(k-1)(1-\epsilon)}$



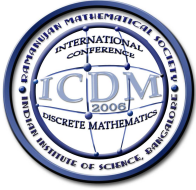
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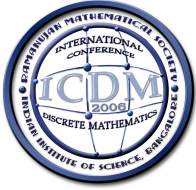
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- If we use $OA(2, k, 2^n)$, the sample points would be n -bit binary vectors and so the scheme can be used for *random pattern built-in self testing of VLSI chips*.
- In some situations, OAs help to completely eliminate random bits used in randomized algorithms so that the resulting algorithm is a deterministic one. This process is called (total) *derandomization*.



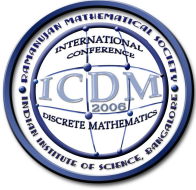
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- There are several more applications of OAs to CS.
- In view of their ubiquity, OAs are now getting recognized as fundamental combinatorial structures (arguably on par with Graphs)
- The relationship between combinatorics and computer science is a mutually beneficial symbiotic one.