Applications of Orthogonal Arrays to Computer Science

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Orthogonal Arrays

An *orthogonal array* $OA_{\lambda}(t, k, v)$ is a $\lambda v^t \times k$ array of symbols from a $v$-set, such that in any $t$ columns, every possible list of $t$ symbols occurs in exactly $\lambda$ rows.
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- If $v = 2$, then these are called **binary orthogonal arrays**.
An Example

Here is a simple $OA(3, 4, 2)$.

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Theory of OAs

For what parameters $t, k, v$ and $\lambda$, do they exist?

For basic theory of OAs, see the recent book on Orthogonal Arrays by Hedayat, Sloane and Stufken. Here we will focus on Applications of OAs to Computer Science.
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- Here we will focus on Applications of OAs to Computer Science.
Applications of OAs

- Threshold Schemes

...and many more.
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- Threshold Schemes
- Authentication Codes
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- Authentication Codes
- Derandomization of Algorithms

Random Pattern Testing of VLSI Chips
Universal Hash Functions
Perfect Local Randomizers
and many more.
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The Model

A particular secret $K \in \mathcal{K}$ is chosen by a special participant called the dealer, $D$. 
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When $D$ wants to share the secret $K$ among the participants in $\mathcal{P}$, he gives each participant some partial information called a *share*.

The shares should be distributed secretly. Let $\mathcal{S}$ be the set of possible values of the shares.
We call such a method of sharing a secret a *perfect \((t, n)\) threshold scheme*, if the following two properties are satisfied.

1. If a subset of participants \(B\) pool their shares, then they can determine the value of \(K\) provided \(|B| = t\).
2. On the other hand, if \(|B| < t\), then they should be able to determine nothing about the value of \(K\).
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- A perfect threshold scheme in which \(|S| = |\mathcal{K}|\), is called an ideal threshold schemes.
Relation to OAs

Theorem: An ideal \((t, n)\) threshold scheme with \(|\mathcal{K}| = \nu\) exists if and only if an \(OA(t, n + 1, \nu)\) exists.
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- First observed by Keith Martin
- Also, independently by Dawson et. al.
- Fairly simple; we shall prove half of it.
Construction

Start with $OA(t, n + 1, v)$. 

First column corresponds to dealer. Remaining columns correspond to the participants. To distribute a specific key $K$, dealer selects a random row of $OA$ such that $K$ appears in the first column. Dealer gives out remaining elements of the row as shares to the participants.
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- Hence, $t - 1$ participants can get no information about the secret.
Authentication Codes - Idea

- Alice wants to communicate to Bob over a public channel.
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- Oscar, the bad guy, can introduce and/or modify messages in the channel.

The purpose is to protect the integrity of the information (and not to provide secrecy). When Bob receives a message from Alice, how can he be sure that the message was really sent by Alice and is not tampered with along the way?
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2. \(A\) is a finite set of *authenticators*. Let \(|A| = l\).
3. \(K\) is a finite set of *keys*.
4. For each \(K \in K\), there is an *authentication rule* \(e_K \in E\). Each \(e_K : S \rightarrow A\).
Authentication Codes - Model

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- Alice sends the message $m = (s, a)$ over the channel.
- When Bob receives $m$, he checks that $a = e_K(s)$ to authenticate.
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- the rows are indexed by authentication rules
- the columns are indexed by source states
- the entry in row $e$ and column $s$ is $e(s)$. 
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- Let $P_{d1}$ be the probability of successfully substituting.
Authentication Codes - Goals

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- Also, minimize the number of authentication rules.

It is not too difficult to show that $P_{d_0} \geq 1/l$ and $P_{d_1} \geq 1/l$, where $l$ is the number of authenticators.
Connection to OAs

**Theorem:** Suppose we have an authentication code for $k$ source states and having $l$ authenticators, in which $P_{d_0} = P_{d_1} = 1/l$. Then

1. $|E| = l^2$, and equality occurs if and only if the authentication matrix is an orthogonal array $OA(2; k; l)$ (with $t = 1$).

2. $|E| = k(l-1) + 1$, and equality occurs if and only if the authentication matrix is an $OA(2; k; l)$ where $t = k(l-1)$. 

ICDM'06—Applications of OAs to CS – p.19
Connection to OAs

Theorem: Suppose we have an authentication code for $k$ source states and having $l$ authenticators, in which $P_{d_0} = P_{d_1} = 1/l$. Then

1. $|\mathcal{E}| \geq l^2$, and equality occurs if and only if the authentication matrix is an orthogonal array $OA(2, k, l)$ (with $\lambda = 1$).
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**Theorem:** Suppose we have an authentication code for \(k\) source states and having \(l\) authenticators, in which \(P_{d_0} = P_{d_1} = 1/l\). Then

1. \(|E| \geq l^2\), and equality occurs if and only if the authentication matrix is an orthogonal array \(OA(2, k, l)\) (with \(\lambda = 1\))
2. \(|E| \geq k(l - 1) + 1\), and equality occurs if and only if the authentication matrix is an \(OA_{\lambda}(2, k, l)\) where

\[
\lambda = \frac{k(l - 1) + 1}{l^2}.
\]
A Glimpse of Proof

Suppose, we use $OA(2, k, l)$ as the authentication matrix. Then, it is easy to see that

Number of authentication rules is $l^2$.

$P_d^0 = 1 = l$ as in each column each authenticator appears exactly $l$ times.

$P_d^1 = 1 = l$ as any ordered pair of authenticators appears exactly once in any two selected columns.
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- $P_{d_1} = 1/l$ as any ordered pair of authenticators appears exactly once in any two selected columns.
Illustration

4 states, 3 authenticators, 9 encoding rules.

<table>
<thead>
<tr>
<th>$s_1$</th>
<th>$s_2$</th>
<th>$s_3$</th>
<th>$s_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
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</tbody>
</table>
Suppose \((s_2, 3)\) is observed by Oscar.

\[
\begin{array}{cccc}
S_1 & S_2 & S_3 & S_4 \\
1 & 1 & 1 & 1 \\
1 & 2 & 2 & 2 \\
1 & 3 & 3 & 3 \\
2 & 1 & 2 & 3 \\
2 & 2 & 3 & 1 \\
2 & 3 & 1 & 2 \\
3 & 1 & 3 & 2 \\
3 & 2 & 1 & 3 \\
3 & 3 & 2 & 1 \\
\end{array}
\]
Illustration

Suppose Oscar wants to substitute $s_4$.

<table>
<thead>
<tr>
<th></th>
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Probabilistic Algorithms

A probabilistic algorithm will be using random bits during the course of its execution unlike a deterministic algorithm. There are two types.

- Las Vegas algorithm may fail to give an answer with probability \( p \), but if it does give an answer, it is correct.
- Monte Carlo algorithm always gives an answer, but the answer may be incorrect with some probability \( q \).
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Generic Monte Carlo Algorithm

Applied to decision problems

Input: $x$

Question: Is $x^2 \leq L$?

A sample point $r \in f_0 \cup f_1 \cup \ldots \cup f_n$ is chosen.

A 0-1 valued deterministic function $f(x; r)$ is computed.

Result declared is $f(x; r)$:

- 0 means No
- 1 means Yes
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Yes-biased Version

If $x \notin L$ then for all $r \in \{0, 1, \ldots, n - 1\}$,
$$f(x, r) = 0.$$
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- If $x \in L$ then the fraction of the values of $r$ for which $f(x, r) = 1$ is at least $1 - \epsilon$.
- So, if the algorithm answers Yes, then it is correct answer.
- A No answer by the algorithm may be wrong.
Exposing the flaw

If the algorithm is run $k$ times, then the error probability is at most $\epsilon^k$ and hence can be made arbitrarily small.
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- Analysis does not reflect reality.
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- Consider an $OA(2, k, n)$.
- This array has $n^2$ rows.
- Generate $2\log n$ true random bits.
- Use them to index a specific row of the OA.
- Run the Monte Carlo Algorithm (k times) using the $k$ elements in the row selected as the sample points.
Calculation of Error Probability

Let $U$ denote the universe of sample points; $|U| = n$. Then $S \subseteq U$ be the set of witnesses; Then $|S| (1 - \frac{1}{n})$. Call a row of the OA a bad row if none of the elements in the row is a witness. Then error probability is simply the probability that a randomly selected row of the OA is bad. Can be shown to be at most $1 + (k - 1)(1 - \frac{1}{n})$ using combinatorial properties of OAs.
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- Call a row of the OA a **bad row** if none of the elements in the row is a witness.
- Then error probability is simply the probability that a randomly selected row of the OA is bad.
- Can be shown to be at most $\frac{\epsilon}{1 + (k-1)(1-\epsilon)}$ using combinatorial properties of OAs.
Remarks

A similar idea was first proposed by Chor and Goldreich under the name *two-point based sampling*. They used a *specific* OA and derived bounds on error probability by using complex techniques.
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The scheme presented is a generalization that works for any OA and yields better error probability.
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- The scheme presented is a generalization that
  - works for any OA.
  - yields better error probability.
  - is analyzable by elementary techniques.
A comparison

<table>
<thead>
<tr>
<th>Name</th>
<th># ran. bits</th>
<th>Error Prob.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Original Scheme</td>
<td>$k \log n$</td>
<td>$\epsilon^k$</td>
</tr>
<tr>
<td>Two-Point Scheme</td>
<td>$2 \log n$</td>
<td>$\frac{\epsilon}{(1-\epsilon)k}$</td>
</tr>
<tr>
<td>OA Scheme</td>
<td>$2 \log n$</td>
<td>$\frac{\epsilon}{1+(k-1)(1-\epsilon)}$</td>
</tr>
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- If we use $OA(2, k, 2^n)$, the sample points would be $n$-bit binary vectors and so the scheme can be used for random pattern built-in self testing of VLSI chips.
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If we use $OA(2, k, 2^n)$, the sample points would be $n$-bit binary vectors and so the scheme can be used for *random pattern built-in self testing of VLSI chips*.

In some situations, OAs help to completely eliminate random bits used in randomized algorithms so that the resulting algorithm is a deterministic one. This process is called (total) *derandomization*.
Concluding Remarks

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- The relationship between combinatorics and computer science is a mutually beneficial symbiotic one.