10. The *Minimum Cover Problem* (MCP) is the following decision problem.

**Input.** Positive integers *N* and *K* and a set *C* = {*x*1, …, *x*m} where *x*k is a subset of {1, …, *N*} for *k* = 1, …, *m*.

**Question.** Is there a subset *C* ′ ⊆ *C* so that |*C* ′| ≤ *K* and the union of all of the sets in *C* ′ is {1, …, *N*}?

**Example** *N* = 7, *K* = 3, *C* = {{2, 4, 6}, {1, 3, 4, 6, 7}, {1, 2}, {4, 6}, {5, 6, 7}}.

The answer to this example is yes. *C* ′ = {{2, 4, 6}, {1, 3, 4, 6, 7}, {5, 6, 7}} fits the requirements.  The cardinality of *C* ′ is 3 ≤ *K* and the union of the sets in *C* ′ is {1, 2, 3, 4, 5, 6, 7}.

 Show that MCP is in NP by describing a polynomial-time evidence checker for it.

Here is an evidence checker for MCP.

Evidence: subset *C* ′ ⊆ *C*

Require: The union of the sets in *C* ′ is {1, …, *N*}

11. The *Exact Cover Problem (XCP)* is the following decision problem.

**Input.** Positive integer *q*, set *C* = {*x*1, …, *xm*} where *xk* is a subset of {1, …, 3*q*} and |*xk*| = 3 for *k* = 1, …, *m*.

**Question.** Does there exists a subset *C* ′ of *C* such that every member of {1, …, 3*q*} occurs in exactly one member of *C* ′ ?

**Example:** *q* = 2, *C* = {{1, 3, 5}, {1, 4, 5}, {1, 2, 6}, {2, 3, 6}, {4, 5, 6}}

Notice that each set in C has 3 members, as required by the problem.  The answer to this example input is yes. Set *C* ′ = {{1, 4, 5}, {2, 3, 6}} satisfies the requirements since every member of {1, 2, 3, 4, 5, 6} occurs in exactly one of the sets in *C* ′.

XCP is known to be NP-complete.

Show that XCP ≤p MCP by giving a polynomial-time mapping reduction from XCP to MCP. Make it clear exactly what your reduction function is.

**Hint.** If input (q, example for XCP, but think of it as an MCP problem.  What modifications do you need to make?

The following is a mapping reduction *f* from XCP to MCP.

$$f(q, C) = (3q q, C)$$

We need to show that $\left(q,C\right)\in XCP \rightarrow \left(3q,q,C\right)\in MCP$ and $\left(3q,q,C\right)\in MCP \rightarrow \left(q,C\right)\in XCP$ where *C* is a collection of 3-member sets.

Say that

1. $C'=\{y\_{1}, …, y\_{m}\}$ is a *cover* of $\{1,…,N\}$ provided $y\_{1} ∪… ∪y\_{m}=\{1,…, N\}$.
2. $C'=\{y\_{1}, …, y\_{m}\}$ is a *partition* of $\{1,…,N\}$ provided every member of $\{1,…, N\}$ occurs in exactly one of $y\_{1}, …, y\_{m}$

Look at the XCP example above. Notice that set C’ is a partition of $\left\{1,…, 3q\right\}$. That observation is all you need. It is no accident that C’ is a partition of $\left\{1,…, 3q\right\}$.

Suppose *C* = {*x*1, …, *xm*}.

$(q, C)$ $\in $ XCP

$\rightarrow $there exists a subset *C’* = {*y*1, …, *yq*} of *C* that is a partition of $\left\{1,…, 3q\right\}$

$\rightarrow $ {*y*1, …, *yq*} covers $\left\{1,…, 3q\right\}$

$\rightarrow $ $\left(3q, q, C\right)\in $ MCP

$\left(3q, q, C\right)\in $ MCP

 $\rightarrow $ there exists a subset *C’* = {*y*1, …, *yq*} of C that is a cover of $\left\{1,…, 3q\right\}$

 $\rightarrow $ {*y*1, …, *yq*} is a partition of $\left\{1,…, 3q\right\}$

 (There are only *q* sets in *y*1, …, *yq* and each of those sets has only 3 members,

 so there are only 3*q* total members in sets *y*1, …, *yq*. There are not enough total

 members for any member to occur more than once and still cover $\left\{1,…, 3q\right\}$.)

$\rightarrow $ $(q, C)$ $\in $ XCP

12. Is there a polynomial time reduction from MCP to XCP? Why or why not?  You must give convincing justification for your answer to get any credit for this question.

Yes. XCP is NP-complete and MCP is in NP. By the definition of an NP-complete problem, there is a polynomial-time mapping reduction from MCP to XCP.