5 Finite-State Machines and Regular Languages

This section looks at a simple model of computation for solving decision problems: a finite-state machine, or FSM.

5.1 Intuitive idea of a FSM

Figure 5-1 shows a diagram, called a *transition diagram*, of FSM M_1. Each circle or double-circle is called a *state*. One of the states, marked by an arrow, is called the *start state*. A state with a double circle is called an *accepting state* and a state with a single circle is called a *rejecting state*.

The arrows between states are called *transitions*, and each transition is labeled by a member of the FSM's alphabet Σ (set $\{a, b\}$ for M_1). For each state q and each member c of Σ, there must be exactly one transition going out of q labeled c.

A FSM is used to recognize a language (a decision problem). To "run" a FSM on string s, start in the start state. Read each character, and follow the transition labeled by that character to the next state. On input "$aabab$", M_1 starts in state 1, then hits states 2, 1, 1, 2, 2, ending in state 2.

The end state determines whether the FSM accepts or rejects the string. Since state 2 is a rejecting state, M_1 rejects "$aabab$". It should be easy to see

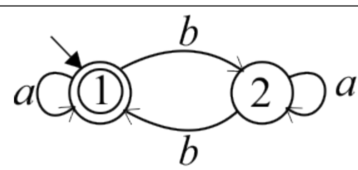

Figure 5-1. Transition diagram of FSM M_1 that recognizes language $\{s \in \{a, b\}^* \mid s \text{ has an even number of } a\text{s}\}$. There are two states. State 1 is the start state. State 1 is an accepting state and state 2 is a rejecting state..

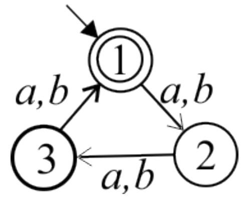

Figure 5-2. Transition diagrams of FSM M_2, which accepts strings whose length is divisible by 3.

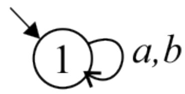

Figure 5-3. Transition diagrams of FSM M_3, which rejects all strings.

that M_1 accepts strings with an even number of as and rejects strings with an odd number of as.

A FSM M with alphabet Σ *recognizes* the set

$$L(M) = \{s \mid s \in \Sigma \text{ and } M \text{ accepts } s\}.$$

For example, $L(M_1) = \{s \mid s \in \{a,b\}^* ands$ has an even number of as$\}$. Figures 5-2 and 5-3 show two finite-state machines M_2 and M_3 with alphabet $\{a, b\}$ where

$$\begin{aligned} L(M_2) &= \{s \mid \ |s| \text{ is divisible by } 3\} \\ L(M_3) &= \{\} \end{aligned}$$

5.2 Designing FSMs

There is a simple and versatile way to design a FSM machine to recognize a selected language L. Associate with each state q the set of strings $\mathrm{Set}(q)$ that end on state q. For example, in machine M_2,

$$\begin{aligned} \mathrm{Set}(0) &= \{s \mid \ |s| \equiv 0 \pmod 3\} \\ \mathrm{Set}(1) &= \{s \mid \ |s| \equiv 1 \pmod 3\} \\ \mathrm{Set}(2) &= \{s \mid \ |s| \equiv 2 \pmod 3\} \end{aligned}$$

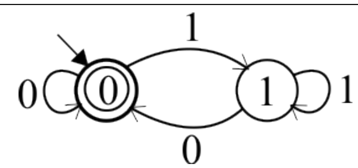

Figure 5-4. A FSM that recognizes even binary numbers. An empty string is treated as 0.

Your goals in designing a FSM that recognizes language L are:

(a) Start by deciding what the states will be and what $\text{Set}(q)$ will be for each state. Make sure that, for each state q, either $\text{Set}(q) \subseteq L$ or $\text{Set}(q) \subseteq \overline{L}$.

(b) Make q be an accepting state if $\text{Set}(q) \subseteq L$ and make q a rejecting state if $\text{Set}(q) \subseteq \overline{L}$.

(c) Draw transitions so that, if $x \in \text{Set}(q)$ and there is a transition from state q to state q' labeled a, then $x \cdot a \in \text{Set}(q')$.

5.2.1 Example: even binary numbers

Figure 5-4 shows a FSM with alphabet {0,1} that accepts all even binary numbers. For example, it accepts "10010" and rejects "1101". $\text{Set}(0) = \{s \in \{0,1\}^* \mid s \text{ is an even binary number}\}$ and $\text{Set}(1) = \{s \in \{0,1\}^* \mid s \text{ is an odd binary number}\}$. The transitions are obvious: adding a 0 to the end of any binary number makes the number even, and adding a 1 to the end makes the number odd.

5.2.2 A FSM recognizing binary numbers that are divisible by 3

Figure 5-5 shows a FSM that recognizes binary numbers that are divisible by 3. For example, it accepts "1001" and "1100", since "1001" is the binary representation of 9 and "1100" is the binary represention of 12. But it rejects "100", the binary representation of 4.

Thinking of binary strings as representing numbers,

$$\begin{aligned}
\mathrm{Set}(0) &= \{n \mid n \equiv 0 \pmod 3\} \\
\mathrm{Set}(1) &= \{n \mid n \equiv 1 \pmod 3\} \\
\mathrm{Set}(2) &= \{n \mid n \equiv 2 \pmod 3\}
\end{aligned}$$

Suppose that m is a binary number that is divisible by 3. Adding a 0 to the end doubles the number, so $m \cdot 0$ is also divisible by 3. Adding a 1 to m doubles m and adds 1. But modular arithmetic tells us that

$$\begin{aligned}
m \equiv 0 \pmod 3 &\rightarrow 2m \equiv 0 \pmod 3 \\
&\rightarrow 2m + 1 \equiv 1 \pmod 3
\end{aligned}$$

so there is a transition from state 0 to state 1 on symbol 1.

5.2.3 Strings containing at least two as and at most one b.

Figure 5-6 shows a FSM that regognizes language

$$\{w \in \{a, b\}^* \mid w \text{ contains at least two } a\text{s and at most one } b\}.$$

The idea is to keep track of the number of as (up to a maximum of 2) and the number of bs (up to a maximum of 2). That suggests that we need nine states: $(0, 0)$, $(0, 1)$, $(0, 2)$, $(1, 0)$, $(1, 1)$, $(1, 2)$, $(2, 0)$, $(2, 1)$ and $(2, 2)$, where the first number is the count of as and the second the count of bs, and 2 means at least 2. The accepting states and transitions should be obvious.

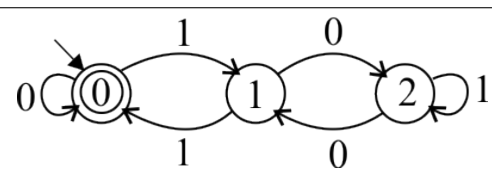

Figure 5-5. A FSM recognizing binary numbers that are divisible by 3. An empty string is treated as 0.

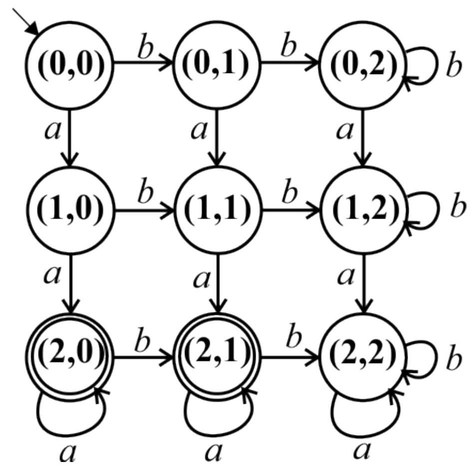

Figure 5-6. A FSM recognizing strings of as and bs with at least two as and at most one b.

5.3 Definition of a FSM and the class of regular languages

The introduction above only shows transition diagrams, and does not adequately say exactly what a FSM is and how to determine the language that it recognizes. This section corrects that with a careful definition of both. The first definition says what a FSM is without saying about what it means to run it on a string.

5.3.1 Definition of a FSM

Definition 5.1. A *finite-state machine* is a 5-tuple $(\Sigma, Q, q_0, F, \delta)$. That is, it is described by those five parts.

- Σ is the machine's alphabet.
- Q is a finite nonempty set whose members are called *states*.
- $q_0 \in Q$ is called the *start state*.
- $F \subseteq Q$ is the set of *accepting states*.

- $\delta : Q \times \Sigma \to Q$ is called the *transition function*.

Most of that should be clear from the transition diagrams that we have looked at. From state q, if you read symbol a, you go to state $\delta(q, a)$. Notice that, because δ is a function, there must be exactly one state to go to from state q upon reading symbol a.

5.3.2 When does FSM M accept string s?

Consider a FSM $M = (\Sigma, Q, q_0, F, \delta)$.

Definition 5.2. If $q \in Q$ and $x \in \Sigma^*$, then $q\,\mathbf{:}\,x$ is defined inductively as follows.

1. $q\,\mathbf{:}\,\varepsilon = q$.
2. If $x = cy$ where $c \in \Sigma$ and $y \in \Sigma^*$ then $q\,\mathbf{:}\,x = \delta(q, c)\,\mathbf{:}\,y$.

The idea is that $q\,\mathbf{:}\,x$ is the state that M reaches if it starts in state q and reads string x. To find that out for string $x = cy$, first find the state $q' = \delta(q, c)$, then finish by finding $q'\,\mathbf{:}\,y$.

Every FSM M has a language $L(M)$ that it recognizes, and the following definition says what that is.

Definition 5.3. $L(M) = \{x \in \Sigma^* \mid q_0\,\mathbf{:}\,x \in F\}$.

That is, M accepts string x if M reaches an accepting state when it is run on x starting in the start state, q_0.

5.3.3 The class of regular languages

Definition 5.4. Language A is *regular* if there exists a FSM M such that $L(M) = A$.

We have see a few regular languages above, including $\{\}$ and the set of binary numbers that are divisible by 3.

5.4 A theorem about $q : x$

Notation $q : x$ satisfies a certain kind of associativity.

Theorem 5.1. $(q : x) : y = q : (xy)$.

Proof. The proof is by induction of the length of x. The introduction to proofs does not cover proof by induction because this is the only such proof that we need. It suffices to

(a) show that $(q : x) : y = q : (xy)$ for all q and y when $|x| = 0$, and

(b) show that $(q : x) : y = q : (xy)$ for an arbitrary nonempty string x, under the assumption (called the *induction hypothesis*) that $(r : z) : y = r : (zy)$ for any state r, string y and string z that is shorter than x.

Case 1 ($|x| = 0$). That is, $x = \varepsilon$. By definition, $q : \varepsilon = q$. So

$$\begin{aligned} (q : x) : y &= q : y \\ &= q : (xy) \end{aligned}$$

because, when $x = \varepsilon$, $xy = y$.

Case 2 ($|x| > 0$). A nonempty string x can be broken into $x = cz$ where c is the first symbol of x and z is the rest.

$$\begin{aligned} (q : x) : y &= (q : (cz)) : y && \\ &= (\delta(q, c) : z) : y && \text{by the definition of } q : (cz) \\ &= \delta(q, c) : (zy) && \text{by the induction hypothesis} \\ &= q : (czy) && \text{by the definition of } q : (czy) \\ &= q : (xy) && \text{since } x = cz \end{aligned}$$

5.5 Closure results

A *closure* result tells you that a certain operation does not take you out of a certain set. For example, $\mathcal{Z}$ is *closed under addition* because the sum of two integers is an integer. $\mathcal{Z}$ is also *closed under multiplication*. But $\mathcal{Z}$ is not closed under division, since $1/2$ is not an integer.

The class of regular languages possesses some useful closure results.

Definition 5.5. Suppose that $A \subseteq \Sigma^*$ is a language. The complement $\overline{A}$ of A is $\Sigma^* - A$.

Theorem 5.2. The class of regular languages is closed under complementation. That is, if A is a regular language then $\overline{A}$ is also a regular language. Put another way, for every FSM M, there is another FSM M' where $L(M') = \overline{L(M)}$. Moreover, there is an algorithm that, given M, finds M'. That is, the proof is constructive.

Proof. Suppose that $M = (\Sigma, Q, q_0, F, \delta)$. Then $M' = (\Sigma, Q, q_0, Q - F, \delta)$. That is, simply convert each accepting state to a rejecting state and each rejecting state to an accepting state.

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Theorem 5.3. The class of regular languages is closed under intersection. That is, if A and B are regular languages then $A \cap B$ is also a regular language. Put another way, suppose M_1 and M_2 are FSMs with the same alphabet Σ. There is a FSM M' so that $L(M') = L(M_1) \cap L(M_2)$. That is, M' accepts x if and only if both M_1 and M_2 accept x. Moreover, there is an algorithm that takes parameters M_1 and M_2 and produces M'.

Proof. The idea is to make M' simulate M_1 and M_2 at the same time. For that, we want a state of M' to be an ordered pair holding a state of M_1 and a state M_2. Recall that the cross product $A \times B$ of two sets A and B is $\{(a, b) \mid a \in A \wedge b \in B\}$.

Suppose that $M_1 = (\Sigma, Q_1, q_{0,1}, F_1, \delta_1)$. and $M_2 = (\Sigma, Q_2, q_{0,2}, F_2, \delta_2)$. Then $M' = (\Sigma, Q', q'_0, F', \delta')$ where

$$\begin{aligned} Q' &= Q_1 \times Q_2 \\ q'_0 &= (q_{0,1}, q_{0,2}) \\ F' &= F_1 \times F_2 \\ \delta'((r, s), a) &= (\delta_1(r, a), \delta_2(s, a)) \end{aligned}$$

State (r, s) of M' indicates that M_1 is in state r and M_2 is in state s. Transition function δ' runs M_1 and M_2 each one step separately. Notice that the set F' of accepting states of M' contains all states (r, s) where r is an accepting state of M_1 and s is an accepting state of M_2. So M' accepts x if and only if both M_1 and M_2 accept x.

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Theorem 5.4. The class of regular languages is closed under union. That is, if A and B are regular languages then $A \cup B$ is also a regular language.

Proof. By DeMorgan's laws for sets,

$$A \cup B = \overline{\overline{A} \cap \overline{B}}.$$

By we know that the class of regular languages is closed under complementatin and intersection.

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