Write clear, readable answers for each of the following. You will be graded down for sloppy work and for work that is difficult to understand.

1. Consider the following equation.

\[ \sum_{i=1}^{n} (i - 1) \cdot 2^i = (n - 2)2^{n+1} + 4 \]  

(a) Show that equation (1) is true for \( n = 1 \) and \( n = 2 \). If you cannot do that, make sure that you understand what equation (1) says.
(b) Using Peano induction, prove that equation (1) holds for every positive integer $n$. Make your proof clear and readable.
2. Suppose that $P(n)$ is the assertion “It is possible to make a total of $n$ dollars using only 3-dollar bills and 5-dollar bills.” (Never mind that there are no 3-dollar bills in the USA.)

(a) Show that $P(8)$, $P(9)$ and $P(10)$ are true.
(b) Using strong induction, show that \( P(n) \) is true for every \( n \geq 8 \). Make your proof clear and readable.
3. What is wrong with the following proof? Address the proof, not the conclusion. Say where the proof makes a mistake, not whether what is being proved is true or false.

**Claim.** For every finite nonempty set \( S \) of hats, all of the hats in set \( S \) have the same size.

**Proof.** Assume that the claim is false. That is, there exists a finite nonempty set of hats whose members do not all have the same size. We call such a set a counterexample.

Choose a smallest counterexample set \( S \). Let \( n = |S| \). Notice that \( n \neq 1 \) since then all hats in \( S \) would have the same size.

So \( n > 1 \). Choose two different proper subsets \( A \) and \( B \) of \( S \), each of size \( n - 1 \), where \( A \cup B = S \). Since \( A \) and \( B \) are smaller than \( S \), they cannot be counterexamples. So all hats in \( A \) have the same size \( s_1 \) and all hats in \( B \) have the same size \( s_2 \).

Choose a member \( h \) of \( A \cap B \). Since \( h \in A \), \( h \) has size \( s_1 \). Since \( h \in B \), \( h \) has size \( s_2 \). So \( s_1 = s_2 \). So all members of \( A \) have size \( s_1 \) and all members of \( B \) also have size \( s_1 \). So all members of \( S \) have size \( s_1 \). That contradicts the assumption that \( S \) is a counterexample.

**Answer:**