Computer Science 2405
April 6, 2020

Happy Monday, April 6.

Recurrences

We are ready to start the last chapter that we will cover, chapter 8 on recurrences.

Imagine an infinite sequence of numbers x_0, x_1, x_2, ... A *recurrence* is a relationships between two, three, four, etc. consecutive values in the sequence.

Example. Recurrence

$$x_n = 2x_{n-1} \qquad (\text{for } n \geq 1)$$

indicates that each value is twice the previous value. That does not completely determine the sequence because it does not specify what x_0 is. But if you say that $x_0 = 1$, then

$$\begin{aligned} x_1 &= 2x_0 \\ &= 2 \\ x_2 &= 2x_1 \\ &= 4 \\ x_3 &= 2x_2 \\ &= 8 \end{aligned}$$

and the rest of the sequence is defined by the recurrence. But if you choose $x_0 = 3$, then

$$\begin{aligned} x_1 &= 2x_0 \\ &= 6 \\ x_2 &= 2x_1 \\ &= 12 \\ x_3 &= 2x_2 \\ &= 24 \end{aligned}$$

Example. Recurrence

$$x_n = x_{n-1} + x_{n-2} \qquad \text{(for } n \geq 2\text{)}$$

gives a relationship between three consecutive values. To define the entire sequence, you need this recurrence plus values of x_0 and x_1. Suppose we choose $x_0 = 0$ and $x_1 = 1$. Then

$$\begin{aligned}
x_2 &= x_1 + x_0 \\
&= 1 \\
x_3 &= x_2 + x_1 \\
&= 2 \\
x_4 &= x_3 + x_2 \\
&= 3 \\
x_5 &= x_4 + x_3 \\
&= 5
\end{aligned}$$

That sequence is called the *fibonacci sequence.*

The values of x_0 and x_1, or of any values that are not defined by the recurrence, are called the initial values or initial conditions.

Chapter is concerned with how to use recurrences as an aid in counting and how to solve recurrences with given initial conditions.

Recurrences as an aid to counting

Read Rosen sections 8.1.1 and 8.1.2.

Example 3 in Rosen section 8.1.2 is concerned with a problem of counting certain bit strings. Let's call a bit-string *good* if it does not have two consecutive 0's, and let a_n be the number of good bit strings of length n. The first few values a_0, a_1, a_2 and a_3 are easy to compute.

- $a_0 = 1$, since there is only one bit string of length 0.
- $a_1 = 2$, since the bit strings of length 1 are 0 and 1.
- $a_2 = 3$ since the good bit strings of length 2 are 01, 10 and 11.
- $a_3 = 5$ since the good bit strings of length 3 are 010, 011, 101, 110 and 111. (000, 001 and 100 are not good.)

It is not obvious how to find a formula for a_n. But we can define a_n in terms of a_{n-1} and a_{n-2}, for $n > 1$, as follows. Every good bit string of length $n > 1$ ends on 1 or 0.

1. If a good bit string ends on 1, then it can be preceded by any good bit string of length $n-1$. For example, the length 3 good bit strings that end on 1 are 011, 101 and 111. You can get them by adding 1 to the end of each of the good bit strings of length 2.
2. If a good bit string of length $n > 1$ ends on 0, then the next-to-last bit must be 1. So the good bit strings of length n that end on 0 are just the good bit strings of length $n-2$ followed by 10. For example, the good bit strings of length 3 that end on 0 are 010, and 110, and you can get them by adding 10 to the end of each good bit string of length 1.

By the sum rule,

$$a_n = a_{n-1} + a_{n-2}$$

whenever $n > 1$. That is the same recurrence as the fibonacci sequence. But the initial condition is $a_0 = 1$ and $a_1 = 2$. Here are the values of a_n for $n = 0, \ldots, 5$.

$$\begin{aligned} a_0 &= 1 \\ a_1 &= 2 \\ a_2 &= 3 \\ a_3 &= 5 \\ a_4 &= 8 \\ a_5 &= 13 \end{aligned}$$

The problem of finding a formula for a_n will need to wait until we look at finding solutions of recurrences.

Example. Suppose that a person can take a step up one stair or up two stairs at once. A step counts just one step, regardless of whether it goes up one stair or two stairs.

Let s_n be the number of different ways to climb n stairs, where the order in which steps are taken matters. Write a recurrence and initial conditions for s_n.

Answer. It should be clear that $s_1 = 1$ and $s_2 = 2$. (You can climb 2 stairs by either taking one two-stair step or by taking two one-stair steps). For $n > 2$, you can either begin with a one-stair step (followed by any way of climbing $n-1$ stairs) or with two-stair step (followed by any way of climbing $n-2$ stairs). By the sum rule,

$$s_n = s_{n-1} + s_{n-2}.$$

Exercise. Let p_n be the number of ways of paying for a stamp that costs n cents if you can only use pennines and nickels and the order in which coins are paid matters. Find a recurrence and initial conditions for p_n for $n \geq 0$.

Answer. For $n < 5$, it is clear that $p_n = 1$; the only option is to pay using only pennies. For $n \geq 5$, you have two options.

1. You can start with a pennie, then finish by paying $n-1$ cents. There are p_{n-1} ways to do that.
2. You can start with a nickel, then finish by paying $n-5$ cents. There are p_{n-5} ways to do that.

So, for $n \geq 5$,

$$p_n = p_{n-1} + p_{n-5}.$$

Exercises

Read Rosen sections 8.1.1 and 8.1.2. Do exercises 2(a,b), 3(a,b), 4(a,b), 5(a,b), 6. Submit them by the end of Thursday, April 9.

Hint for question 4. A ternary string of length $n > 2$ that has two consecutive 0's can begin with 1 or 2, followed by any ternary string of length $n-1$ that has two consecutive 0's. If it starts with 0, then its first two characters can be 00, 01 or 02. What can 00 be followed by? What can 01 be followed by?

Hint for question 6. Write two recurrences. Define a_n to be the number of bit strings of length n that have an even number of 0's. Define b_n to be the number of bit strings of length n that have an odd number of 0's. Give initial conditions and a recurrence for each of a_n and b_n.

Submit a Microsoft Word attachement. Do not write the answers in the email body. The file name must begin with your last name, followed by your first name. The email subject must be CSCI 2405 homework 5.1.