Computer Science 2405
March 30, 2020

Happy Monday, March 30.

Since we are doing the homework in small installments, nothing will be due on April 3. Exam 4 is April 13.

Work today's exercises and submit by the end of Wednesday, April 1.

The binomial theorem

Combinations come up all over mathematics, and one reason for that is the binomial theorem. Notice that

$$(x+y)^2 = x^2 + 2xy + y^2.$$

Making the coefficients all explicit,

$$(x+y)^2 = (1)x^2 + (2)xy + (1)y^2.$$

The sequence of coefficients is 1, 2, 1, which is row 2 of Pascal's triangle. Also notice that

$$(x+y)^3 = (1)x^3 + (3)x^2y + (3)xy^2 + (1)y^3.$$

The sequence of coefficients is 1, 3, 3, 1, which is row 3 of Pascal's triangle. Notice that the sum of the exponents of x and y is 3 in all of the terms.

That generalizes to any positive integer power of $(x+y)$.

Binomial Theorem. Suppose x and y are variables and n is a positive integer. Then

$$\begin{aligned}(x+y)^n &= \sum_{j=0}^{n} \binom{n}{j} x^{n-j} y^j \\ &= \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \cdots + \binom{n}{n-1}xy^{n-1} + \binom{n}{n}y^n.\end{aligned}$$

Notice that the sum of the exponents of x and y is n in all of the terms.

Example. If you multiply out $(x+y)^5$, what do you get?

Answer. Row 5 of Pascal's triangle is

```
1   5   10   10   5   1
```

So

$$(x+y)^5 = x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5.$$

Corollary. Suppose that n is a positive integer. Then

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n.$$

Proof. According to the binomial theorem,

$$\begin{aligned} (1+1)^n &= \sum_{k=0}^{n} \binom{n}{k} 1^{n-k} 1^k \\ &= \sum_{k=0}^{n} \binom{n}{k} \end{aligned}$$

Remark. Suppose that S is a set of size n. We have seen that S has 2^n subsets. For example, the subsets of $\{a, b\}$ are $\{\}$, $\{a\}$, $\{b\}$ and $\{a, b\}$, and there are 2^2 of them. S has one subset of size 0, two subsets of size 1 and one subset of size 2, and $1 + 2 + 1 = 4 = 2^2$.

The corollary above is really just counting subsets in two different ways. There are $\binom{n}{k}$ subsets of S of size k. Adding up the number of subsets by size gives

$$\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n}$$

which must surely give 2^n, since that is the total number of subsets.

Exercise

Do exercise 19 in homework set 4.

Permutations with repetitions

Suppose that S is a set of size n. How many ways are there to write a sequence of r members of S if repetitions are allowed?

We worked that out earlier. Because repetitions are allowed, there are n options for each of the r choices. So there are n^r sequences.

Read Rosen 6.5.2.

Combinations with repetitions

Combinations with repetitions allowed take more thought. Let's think about an example. How many ways are there to select 6 values from set $\{a, b, c, d\}$ if repetitions are allowed and order does not matter? Some of the possibilities are: *aaaaaa*, *abcddd*, *aabccc* and *aabbcc*. Notice that *dddcab* is the same as *abcddd* since order does not matter. We can handle that by always writing the combination in alphabetical order.

The key to counting combinations with repetition allowed lies in how to represent the combinations. Let's put a bar between sections of different values. For example, *abcddd* becomes $a|b|c|ddd$. We will always write 3 bars, even when some of the sections are empty. For example, *aabddd* is written $aa|b||ddd$ and *bbbbbb* is written $|bbbbbb||$.

Since values a, b, c and d are written in alphabetical order, there is no need to show them explicitly. Let's replace each one by an asterisk. For example, *aabddd* is `**|*||***`. There are 4 sections, separated by 3 bars. Since the first section has two asterisks, there are two a's in this combination with repetition. Notice that the third section has no asterisks, meaning that there are no c's.

How many strings of 6 asterisks and 3 bars are there? Just choose locations for the bars. All of the others must be asterisks. There are $6 + 3 = 9$ characters in a string of 6 asterisks and 3 bars, and there are $\binom{9}{3}$ ways of choosing placements of bars in the string. So there are $\binom{9}{3}$ ways to choose 6 things from a set of size 4 when repetitions are allowed and order does not matter.

Now let's generalize to choosing r values from a set of size n with repetitions allowed and where order does not matter. That is equivalent to choosing a string of r asterisks and $n - 1$ bars.

Theorem. There are $\binom{n+r-1}{r}$ (or, equivalently, $\binom{n+r-1}{n-1}$) ways to choose r values from a set of size n when repetitions are allowed and order does not matter.

Example. How many ways are there to select 5 cookies at a cookie shop that offers 6 different varieties of cookies, where you can select as many as you want of each variety and the order in which you select them does not matter. Assume that at least 5 cookies of each variety are available.

Answer. There are $\binom{5+6-1}{5} = \binom{10}{5}$ ways to choose the cookies.

$$\binom{10}{5} = \frac{(10)(9)(8)(7)(6)}{(5)(4)(3)(2)}$$

$$
\begin{aligned}
&= \frac{(9)(8)(7)(6)}{(4)(3)} \\
&= (9)(4)(7) \\
&= 252.
\end{aligned}
$$

Exercises

Do exercises 34, 35, 36 in homework set 4.

Read Rosen 6.5.3.

Permutations with indistinguishable objects

How many permutations of string **aabb** are there?

```
aabb
abab
abba
baab
baba
bbaa
```

Notice that there are fewer than 4! permutations, even though *aabb* has length 4, because there is no way to distinguish one a from another, or to distinguish one b from another.

To count the number of permutations of *aabb*, let's imagine for the moment that the a's are distinguishable and the b's are distinguisable. For this simple example, we can use a and A for the two occurrences of a and use b and B for the two occurrences of b. Here are the 24 permutations of $aAbB$.

```
aAbB
aABb
AabB
AaBb

abAB
```

```
aBAb
AbaB
ABab

abBA
aBbA
AbBa
ABba

baAB
BaAb
bAaB
BAab

baBA
BabA
bABa
BAba

bBaA
BbaA
bBAa
BbAa
```

Each group of symbols (such as a and A) that correspond to one kind of item (a) can be permuted in any way. Since there are two ways to permute aA and two ways to permute bB, each permutation of string $aabb$ corresponds to 2x2 = 4 ways of permuting $aAbB$. The division rule comes into play: there are 4! permutations of $aAbB$ and 4!/(2x2) = 6 permutations of $aabb$.

Permutations with indistinguishable values
Suppose that $x_1, x_2, \ldots, x_n$ is a sequence that has c_1 copies of y_1, c_2 copies of y_2, ... , c_m copies of y_m. There are

$$\frac{n!}{c_1!\,c_2!\,\cdots c_m!}$$

permutations of $x_1, x_2, \ldots, x_n$.

Example. In the case of $aabb$, the sequence length is $n = 4$, there are $m = 2$ different values, with $c_1 = 2$ copies of a and $c_2 = 2$ copies of b.

So there are

$$\frac{4!}{(2!)(2!)} = 6$$

permutations of *aabb*.

Example. How many permutations are there of BABBLE?

Answer. There are 4 different letters, A, B, E and L. There is 1 copy of A, 3 copies of B, one copy of E and one copy of L. So there are

$$\frac{6!}{(1!)(3!)(1!)(1!)} = 120$$

permutations of BABBLE.

Exercises

Read Rosen section 6.5.4.

Do exercises 15, 30, 31, 37 in homework set 4.