Answer all questions in a separate document and email that document to abrahamsonk@ecu.edu as an attachment. Use subject CSCI 2405 final exam. You can attach graphics files for hand-written answers, but be sure that they are readable. Make all file names, including graphics files, begin with your last name, followed by your first name. Write your name in your answer file. Make your answers clear, concise and precise.

1. Pascal’s identity states that

\[
\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}
\]

for all \( n \geq 0 \) and \( k \geq 0 \) where \( n \geq k \). You will find that fact useful in this question.

Consider the following equation.

\[
\sum_{k=2}^{n} \binom{k}{2} = \binom{n+1}{3}
\]  \hspace{1cm} (1)

(a) Show that equation (1) holds for \( n = 2 \).

(b) Suppose \( P(n) \) is the assertion that equation (1) holds for a particular value \( n \). By substituting in equation (1), express \( P(n+1) \).

(c) Show, by induction on \( n \), that equation (1) is true for all integers \( n \geq 2 \), by showing that \( P(n) \Rightarrow P(n+1) \) for all \( n \geq 2 \).

2. Show, by induction on \( n \), that 9 is a divisor of \( n^3 + (n+1)^3 + (n+2)^3 \) for every positive integer \( n \).

3. Suppose that \( G \) is a connected planar simple graph with 40 vertices, where every vertex has degree 3. Into how many regions does a planar embedding of \( G \) divide the plane?
4. It is known that every planar graph can be colored with four colors, where no two adjacent vertices have the same color. Is it true that every nonplanar graph requires more than 4 colors? If so, explain why. If not, give an example of a nonplanar graph that can be colored with no more than 4 colors.

5. Suppose that $G$ and $H$ are simple graphs. What is the definition of an isomorphism from $G$ to $H$?

6. Suppose that simple graphs $G$ and $H$ are isomorphic, and suppose that $G$ has a subgraph of 5 vertices that is complete, meaning that each of those 5 vertices is adjacent to each other one. Show that $H$ also has a complete subgraph of 5 vertices by using the definition of an isomorphism, showing exactly how to find the complete subgraph in $H$ from the complete subgraph in $G$, and showing that the chosen subgraph of $H$ must be complete. Do not give an argument that relies only on intuition.

7. How many bit strings of length 8 contain exactly 5 1’s?

8. How many bit strings of length 9 either start with 11 or end with 11?

9. Prove that, given any set $S$ of 10 positive integers not exceeding 50, there exist at least two different 5-element subsets of $S$ that have the same sum.

10. How many ways are there to put 10 identical balls into 5 different (and distinguishable) baskets?

11. How many different arrangements (orders) are there of the numbers \{1, 2, 3, 4, 5, 6, 7\} where 1 and 2 are consecutive, with either one first? That is, the arrangement can have the form \ldots 12\ldots or \ldots 21\ldots, where the 1 and 2 can occur anywhere in the sequence.

12. How many arrangements are there of the letters in “banana”?

13. Suppose that function $f(n)$ satisfies recurrence

\[ f(n) = 16f(n/2) + n^4 \]

whenever integer $n \geq 2$ is a power of 2. Give a closed form solution for $f(n)$ that holds to within a constant factor. (That is, find an expression $E$ involving $n$ so that $f(n)$ is $\Theta(E)$.)
14. Suppose that function $f(n)$ satisfies recurrence

$$f(n) = 16f(n/2) + n^3$$

whenever integer $n \geq 2$ is a power of 2. Give a closed form solution for $f(n)$ that holds to within a constant factor. (That is, find an expression $E$ involving $n$ so that $f(n)$ is $\Theta(E)$.)

15. Suppose sequence $a_0, a_1, a_2, \ldots$ satisfies recurrence

$$a_n = 4a_{n-1} - 3a_{n-2}$$

for $n > 1$.

(a) What is the characteristic equation of recurrence (2)?
(b) What are the solution(s) to the characteristic equation?
(c) What is the general form of a solution of recurrence (2)? Express it using names for arbitrary constants.

16. Suppose sequence $a_0, a_1, a_2, \ldots$ satisfies recurrence

$$a_n = 8a_{n-1} - 16a_{n-2}$$

for $n > 1$.

(a) What is the characteristic equation of recurrence (3)?
(b) What are the solution(s) to the characteristic equation?
(c) What is the general form of a solution of recurrence (3)? Express it using names for arbitrary constants.