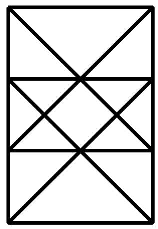
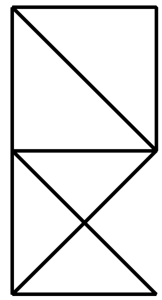
Discrete Mathematics **Name: Answers**
Exam 1 **February 18, 2009**

1. For each of the figures shown below, please indicate whether they have an Euler path, an Euler circuit, both or neither.

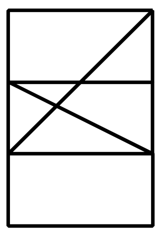
This one has more than two odd vertices, so NEITHER.

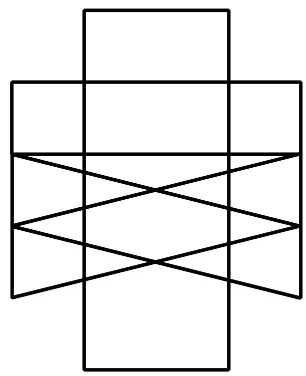




This one has two odd vertices, so PATH

This one has two odd vertices, so PATH





This one has no odd vertices, so PATH and CIRCUIT

2. When will a graph have an Euler circuit? *When all of its vertices have even degree.*

3. What about a graph will guarantee that it has neither an Euler circuit nor an Euler path? *If it has more than two odd vertices.*

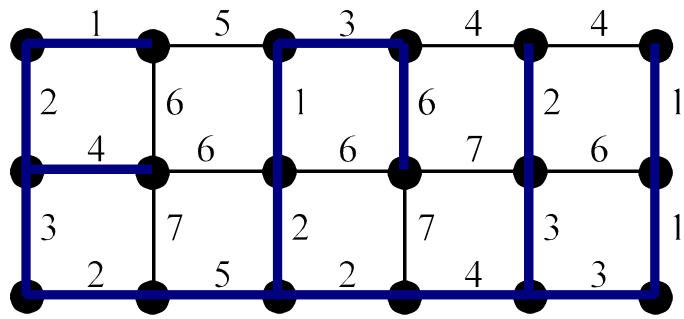
4. G is a graph that has 20 vertices, and all of them have degree five. How many edges does this graph have? *Explain your answer. It will have* $20 \times 5 / 2 = 50$ *edges*

5. Give a proof that a graph cannot have exactly one vertex of odd degree, regardless of how many vertices it has of even degree.
Proof 1: *Since the sum of the degrees is equal to twice the number of edges, it follows that the sum of the degrees must be even. But if there was exactly one odd vertex, this sum would be odd. Contradiction.* ■
Proof 2: *If you construct the graph by starting with vertices only, and no edges, then you start with a graph in which all vertices have even degree. Then add the edges one at a time. Each time you add an edge, you change the parity of the two vertices incident with the edge. Thus each time you add an edge, the number of odd vertices must either go up by 2, down by 2 or stay the same. This implies that the number of odd vertices must remain even.* ■

6. Prove that if T is a tree and u and v are two vertices in T, then there is a path from u to v in T.
Proof: *If T is a tree, then it is connected and has no cycles, by definition of tree. And by definition of connected, there will be a path between any pair of vertices in T, including u and v.*

7. Suppose that G is a tree and uv is an edge in G connecting vertices u and v. Suppose we remove the edge uv from G. Prove that the resulting graph will not have a path from u to v.
Proof: *(By contradiction.) Suppose there was a path from u to v in the resulting graph. Then if we put the edge uv back, we will have found a cycle in the original graph G. But G is a tree, and must therefore have no cycles. Contradiction.* ■

8. Find a minimum weight spanning tree in the graph below: (Two copies are given. Please cross out the one that you don't want graded.)



9. Describe in words the method you used to find the minimum weight spanning tree in the previous problem. (Don't merely say "Kruskal's algorithm" or "Prim's algorithm." Say a few words about how you use the algorithm.)
Kruskal's Algorithm: *At each step select the edge of minimum weight that will not create a cycle. Stop when you have a spanning tree.*
Prim's Algorithm: *Select a start vertex, v, and call this your tree so far. At each step, select the edge incident with your tree so far having minimum weight, but without creating a cycle. Stop when your tree contains all vertices of the graph.*

10. Prove that $p \wedge \neg q$ is logically equivalent to $\neg(q \vee \neg p)$. *The boxed columns below are identical.*

p	**q**	$\neg q$	$p \wedge \neg q$	$\neg p$	$q \vee \neg p$	$\neg(q \vee \neg p)$
t	t	f	f	f	t	f
t	f	t	t	f	f	t
f	t	f	f	t	t	f
f	f	t	f	t	t	f

11. Write the contrapositive of each of these implications:
   a. If you give me ten dollars, then I'll give you a USB drive
      If I don't give you a USB drive, then you won't have given me ten dollars
   b. If p is prime, then $p + 1$ is composite
      If p+1 is not composite, then p is not prime.

12. Prove that $p \rightarrow (p \vee q)$ is true regardless of the truth values of p and q. *If p is true, then p OR q is true, so the implication is true. If p is false, then the implication will be true regardless of the value of p OR q.*

13. Let A = {1, 2, 3, 4, 5}, B = {1, 3, 5, 7, 9} and C = {1, 2, 3, 4}. Compute each of the following:

a. $A \cup C$

{1, 2, 3, 4, 5}

b. $B - C$

{5, 7, 9}

c. $C \cap (B \cup A)$

$B \cup A$ = {1, 2, 3, 4, 5, 7, 9}, and so

$C \cap (B \cup A)$ = {1, 2, 3, 4}

d. $A - A$

$\varnothing$

e. $A \cap \varnothing$ ($\varnothing$ is the empty set.)

$\varnothing$

f. $B \cup \varnothing$

Just B = {1, 3, 5, 7, 9}

g. $A \cup B \cup C$

{1, 2, 3, 4, 5, 7, 9}

14. Prove or disprove: If p is odd and prime, then $p^2 + 2$ is prime.
Disprove with a counterexample: *When p = 5, p2 + 2 = 27 which is not prime.*

15. Prove or disprove: If *G* is a graph with all vertices of even degree, then *G* has an even number of edges.
Disprove with a counterexample: *If we consider three vertices connected in a triangle with three edges, we get a counter-example.*

16. Prove or disprove: If the average of three integers is 10, then the sum of those integers must be even.
Proof: *Let x, y and z be the integers. Then $(x + y + z) / 3 = 10$, meaning $x + y + z = 30$, which is even.*

17. Prove or disprove: A tree on ten vertices cannot have a vertex of degree five.
Disprove with a counterexample: