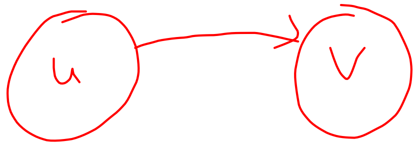
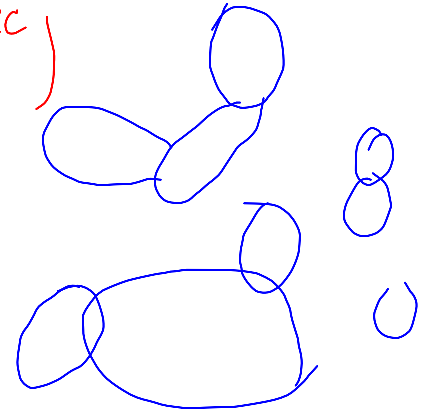
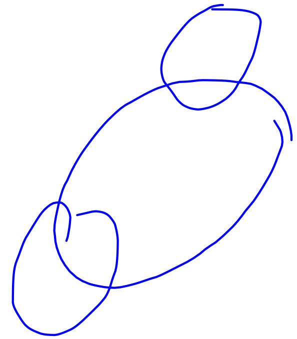
Strongly Connected Components

The "Strongly Connected Component Graph" (G^{SCC}) of a digraph G has a vertex for each SCC of G, and an arc from component u to component v



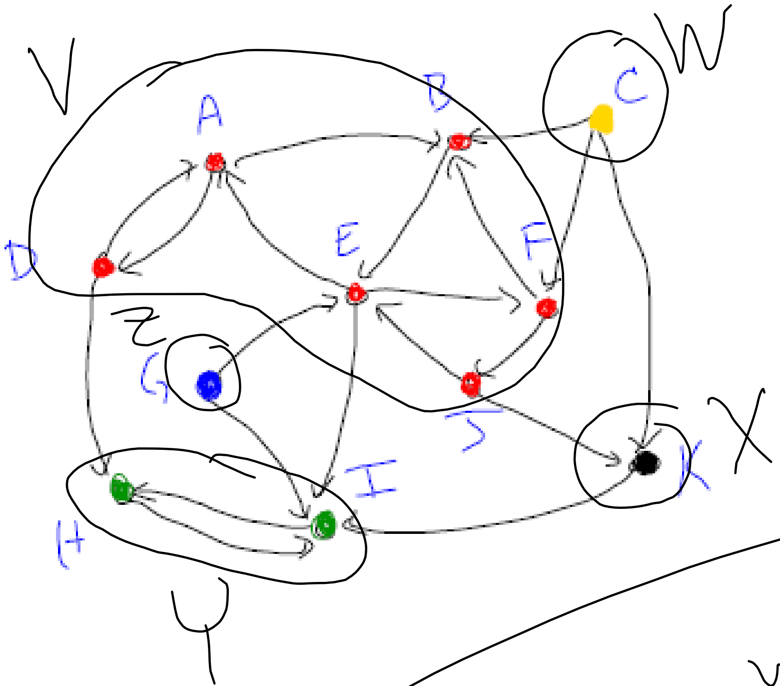
if G has an arc from some vertex in u to some vertex in v.

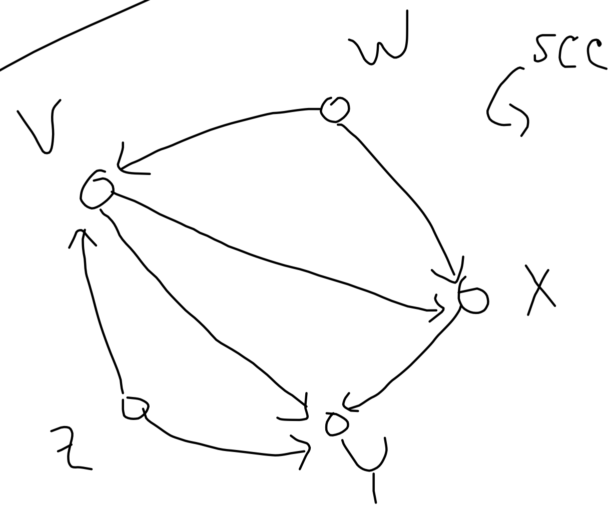




G^{SCC} has 5 vertices

Can the SCC graph ever contain a cycle?

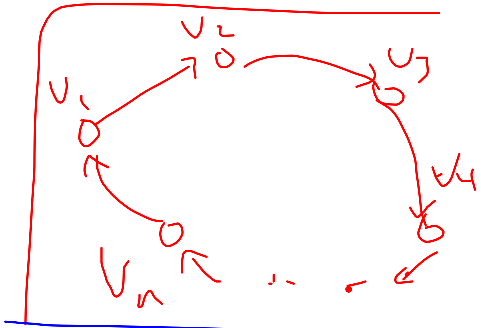




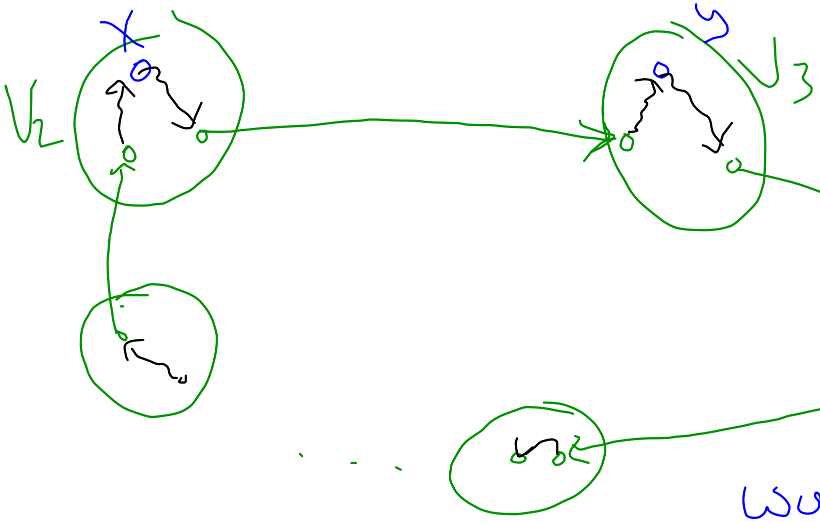
Def: A __dag__ is a digraph that has no cycles (directed acyclic graph).

thm: For any digraph D, D^{scc} is a dag.

Proof: Suppose D^{scc} had a cycle.



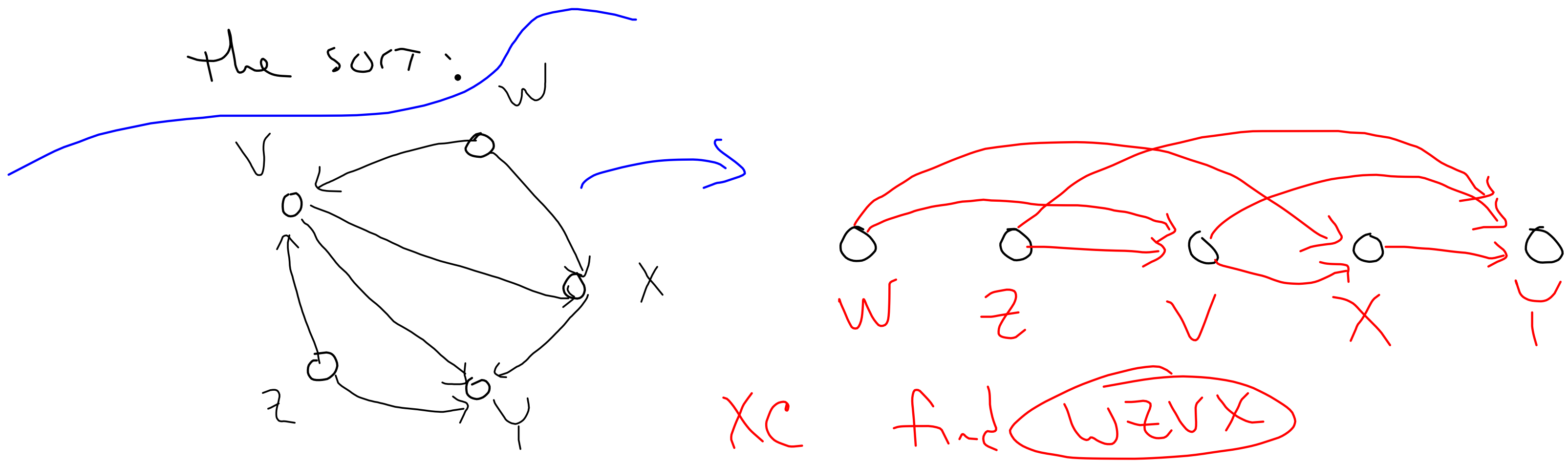
$$v_1 \to v_2 \to \cdots \to v_n \to v_1.$$



Then these components ~~would~~ should have been a single SCC, and the original $v_1, v_2, \ldots$ would not have been maximal.

Fact: Adding an edge to a digraph can never increase the # of SCC's. (It can decrease it arbitrarily, or leave it the same.)

Def: A *topological sort* of a dag is an ordering of the vertices of the dag so that all edges go "from left-to-right," that is, from lower-to-higher vertices in the sort:

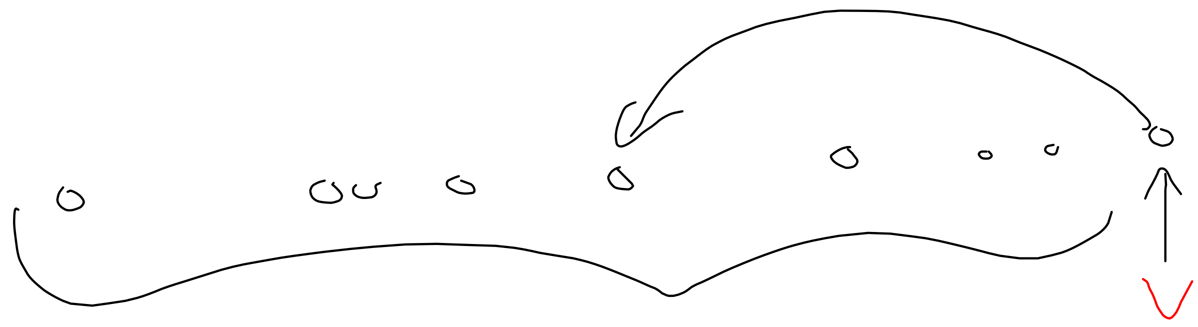


XC find WZVX

~~Theorem~~ Fact: If a digraph has a cycle, then it does not have a topological sort.

Proof: Let C be the set of vertices on the cycle.

C:



Let v be the rightmost vertex in an alleged topological sort. v must have an arc in the cycle back to some earlier vertex (to the left.) Contradiction. ■

<u>Thm (?):</u> Do a DFS on a dag. The ordering of the vertices, highest-to-lowest by finishing times, is a topological sort.