Let's prove: If $T(0)=1, T(1)=0$,

$T(n) = 2T(n-1) + T(n-2)$ for $n \geq 2$,

Is $T(n) = \cancel{\Theta(n^2)}$? $O(3^n)$?

Assume it's true for $k < n$: $T(k) \leq c \cdot 3^k$

Show it's true for n. Show, $T(n) \leq 3^n \cdot c$

$$3^n = 3 \cdot 3 \cdot 3^{n-2} = 9 \cdot 3^{n-2}$$

$$\frac{1}{9} \cdot 3^n = 3^{n-2}$$

$$\begin{aligned} T(n) &= 2T(n-1) + T(n-2) \\ &\leq 2 \cdot c \cdot 3^{n-1} + c \cdot 3^{n-2} \\ &= 2 \cdot c \cdot \frac{1}{3} \cdot 3^n + c \cdot \frac{1}{9} \cdot 3^n \\ &= 3^n \left(\frac{2}{3} \cdot c + \frac{1}{9} c\right) = 3^n \cdot \left(\frac{7}{9} \cdot c\right) \leq 3^n \cdot c \end{aligned}$$

By induction, $T(n) \leq 3^n \cdot c$ for any $c > 0$

$\longrightarrow T(n) = o(3^n)$ (Bonus discovery)

Would $T(n) \le c \cdot 2^n$ have worked?

$$T(n) = 2T(n-1) + T(n-2)$$

$$\le 2 \cdot c \cdot 2^{n-1} + c \cdot 2^{n-2}$$

$$= c \cdot 2^n + \frac{1}{4} \cdot c \cdot 2^n$$

$$= 2^n\left(c + \frac{1}{4}c\right)$$

$$= 2^n\left(\frac{5}{4}c\right) \ \underline{\text{not}} \le 2^n \cdot c$$

$c \cdot 2^n \cdot \left(1 + \frac{1}{4}\right)$

Want $\le c \cdot 2^n$

In fact, we have shown that, for any value of c, our T(n) will be greater than c*2^n. So we have a little omega(T(n)) bound.

Let's find the right 2^n z^n 3^n

$T(n) \doteq c \cdot z^n$ ("$\doteq$" is "sorta =")

$T(n) = 2T(n-1) + T(n-2)$

$z^n = 2z^{n-1} + z^{n-2}$

divide by z^{n-2}

$$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

(quadratic formula, Ralph)

$z^2 = 2z + 1$

$z^2 - 2z - 1 = 0$

$$z = \frac{2 \pm \sqrt{4+4}}{2} = \frac{2 \pm \sqrt{8}}{2} = 1 \pm \sqrt{2}$$

$1 - \sqrt{2} < 0$, so that's not right

$z = 1 + \sqrt{2}$ is right

In fact, you can prove that

$$T(n) = \Theta\left((1+\sqrt{2})^n\right)$$

- Prove $T(n) = O\left((1+\sqrt{2})^n\right)$

  (Just as we did with 3^n)

- Prove $T(n) = \Omega\left((1+\sqrt{2})^n\right)$

  (Also same as our 3^n proof, but do $\geq$ instead of $\leq$)

$$T(n) = 3T\left(\left\lfloor \frac{n}{2} \right\rfloor\right) + 2$$

Let's prove $T(n) = \Omega(n^2)$

We don't have $\Omega(n^2)$

Assume $T(k) \geq c \cdot k^2$ for $k < n$.

Show $T(n) \geq c \cdot n^2$.

$$\frac{n-1}{2} \leq \left\lfloor \frac{n}{2} \right\rfloor \leq \frac{n}{2}$$

$$x - 1 \leq \lfloor x \rfloor \leq x$$

$$T(n) = 3T\left(\left\lfloor \frac{n}{2} \right\rfloor\right) + 2$$

$$\geq 3 \left\lfloor \frac{n}{2} \right\rfloor^2 \cdot c + 2$$

$$\geq 3\left(\frac{n-1}{2}\right)^2 \cdot c + 2$$

$$= \frac{3}{4}(n^2 - 2n + 1) \cdot c + 2$$

$$= cn^2\left(\frac{3}{4} - \frac{6}{4n} + \frac{3}{4n^2}\right) + 2$$

for large n, this is $\geq \frac{3}{4} \cdot c \cdot n^2$

~~$\geq$~~ BUT not $\geq c \cdot n^2$

HW: Show $T(n) = \Omega(n)$

(or not . . .)