The Standing Bosses Game

Line up some players, in chairs, facing the class

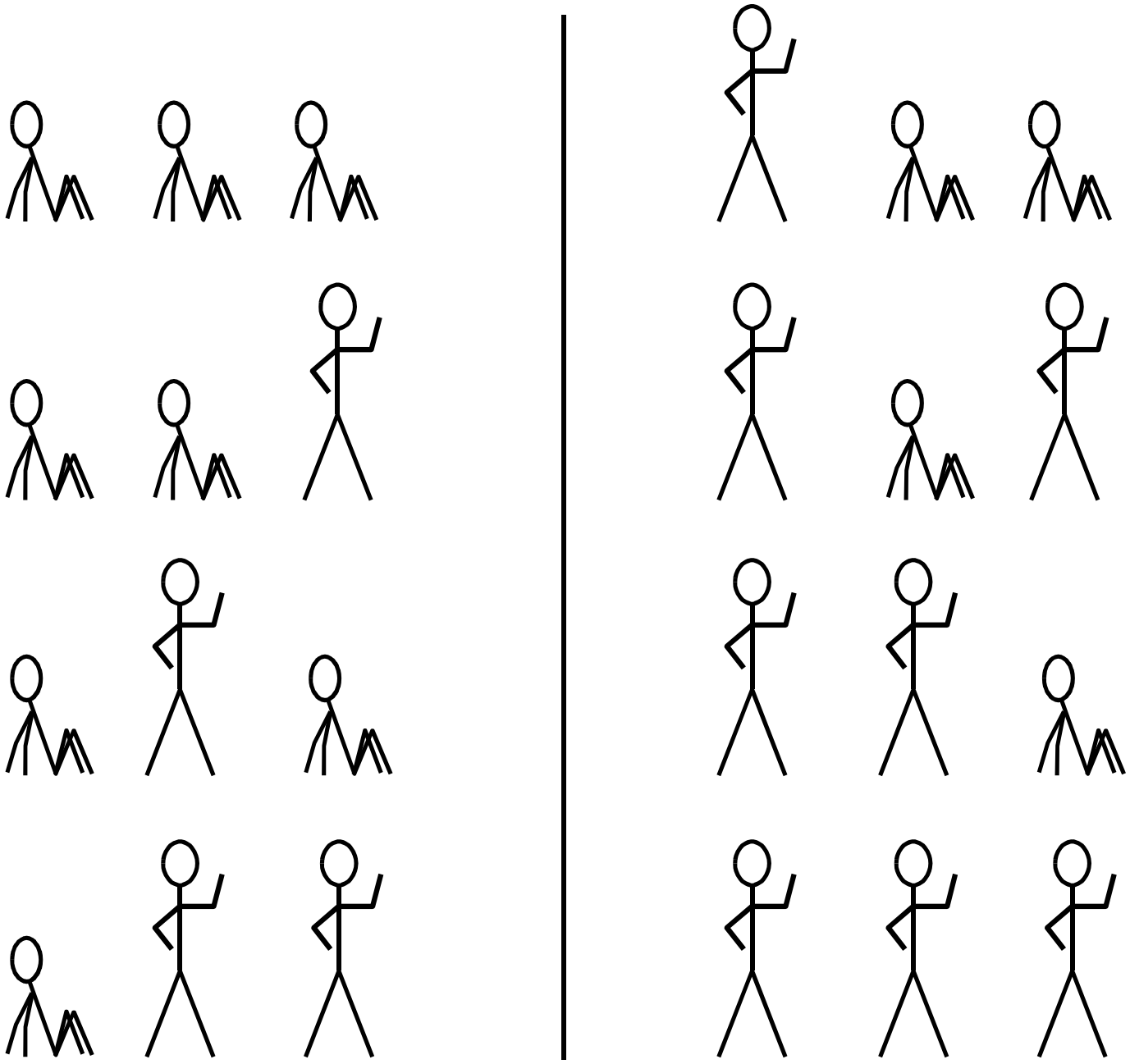
- The rightmost sitting player (from the class's point of view) is the **boss**

- The **boss**'s job is to stand up, and to tell all standing players to his right (from the class's point of view) to sit down

The game is repeated for as long as possible

What happens?
Will this game ever end?
What will the "end" look like?

The Game with 3 People



How long would the game last with 4 people?

How long would the game last with 5 people?

Symbolic Representation of the Game

0 0 0	1 0 0
0 0 1	1 0 1
0 1 0	1 1 0
0 1 1	1 1 1

The Game with 3 People

000
001
010
011
100
101
110
111

The Game with 4 People

0000
0001
0010
0011
0100
0101
0110
0111
1000
1001
1010
1011
1100
1101
1110
1111

The Game with 5 People

00000	10000
00001	10001
00010	10010
00011	10011
00100	10100
00101	10101
00110	10110
00111	10111
01000	11000
01001	11001
01010	11010
01011	11011
01100	11100
01101	11101
01110	11110
01111	11111

Okay. What patterns do you see?

Interesting Observation Number One

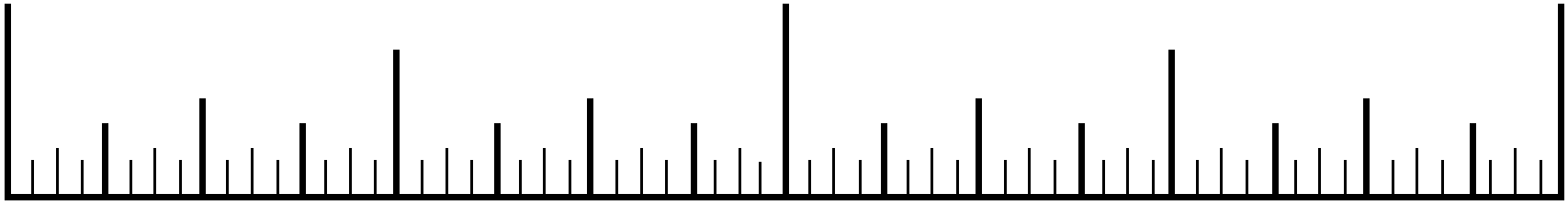
8	4	2	1	Total
0	0	0	0	
0	0	0	1	
0	0	1	0	
0	0	1	1	
0	1	0	0	
0	1	0	1	
0	1	1	0	
0	1	1	1	
1	0	0	0	
1	0	0	1	
1	0	1	0	
1	0	1	1	
1	1	0	0	
1	1	0	1	
1	1	1	0	
1	1	1	1	

The Game secretly encodes the counting numbers!

The Game with 5 People

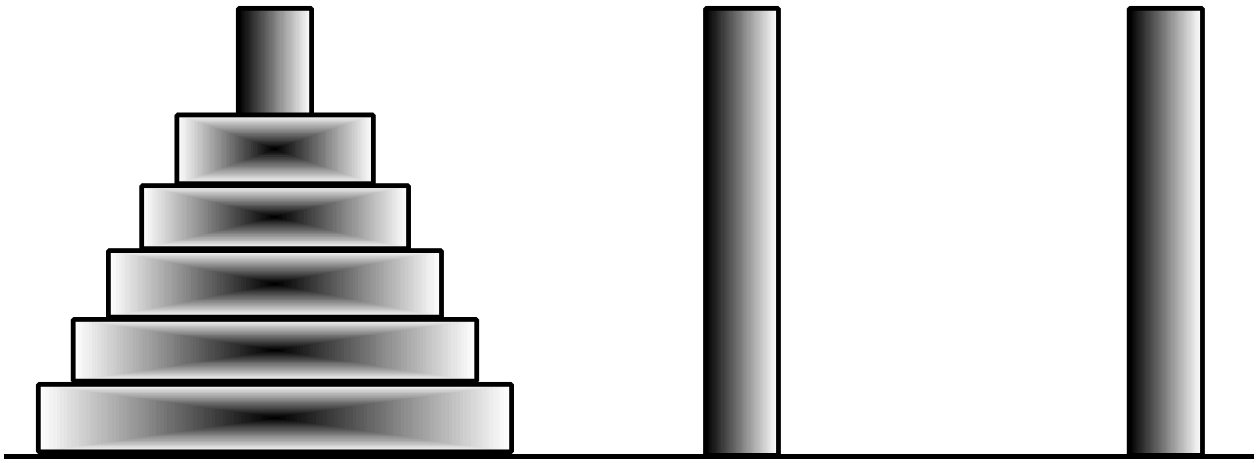
00000
00001
00010
00011
00100
00101
00110
00111
01000
01001
01010
01011
01100
01101
01110
01111
10000
10001
10010
10011
10100
10101
10110
10111
11000
11001
11010
11011
11100
11101
11110
11111

Interesting Observation Number Two
or
Who's the Boss?



You've seen this pattern before!

Interesting Observation Number Three
The Tower of Hanoi Puzzle



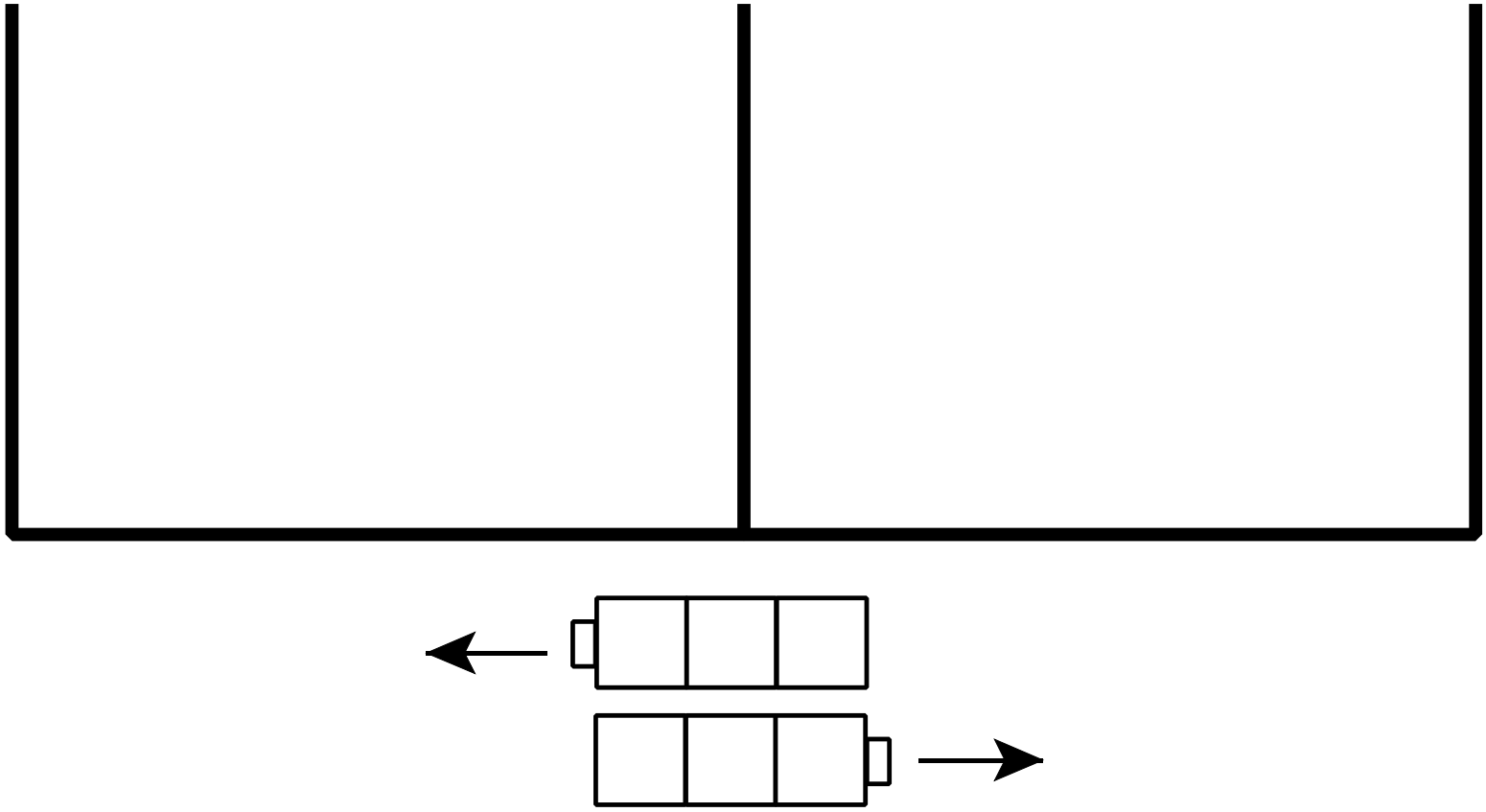
A set of different-sized discs are stacked on one of three poles. The stack must be moved to another pole, according to the following rules:

- You may move only one disc at a time.
- During the transfer, you may never put a disc on top of a smaller disc.

How can one move all the discs from the start pole to another pole, and how many moves will it take?

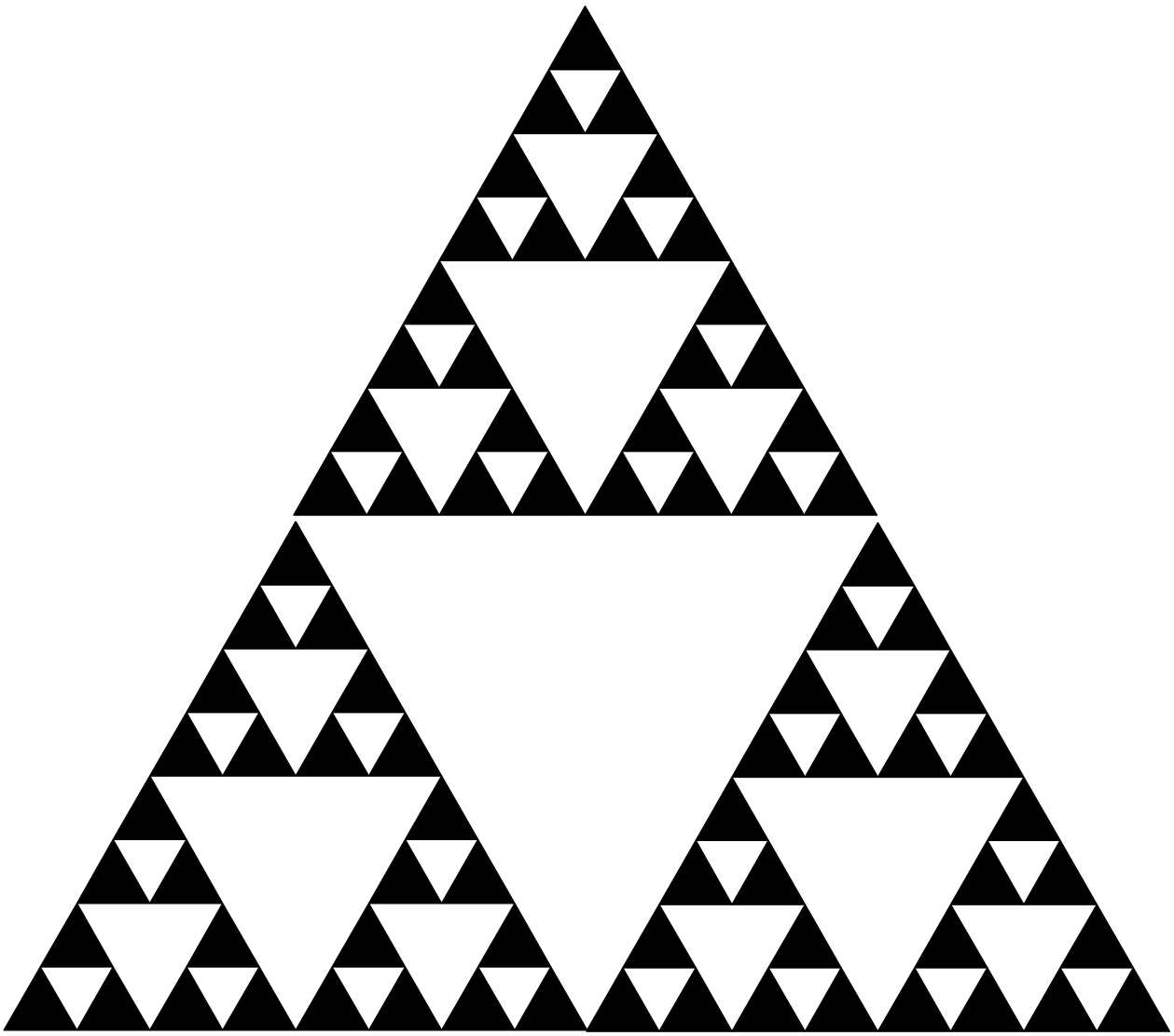
Moving the Blocks

small, medium, small, large, small, medium, small

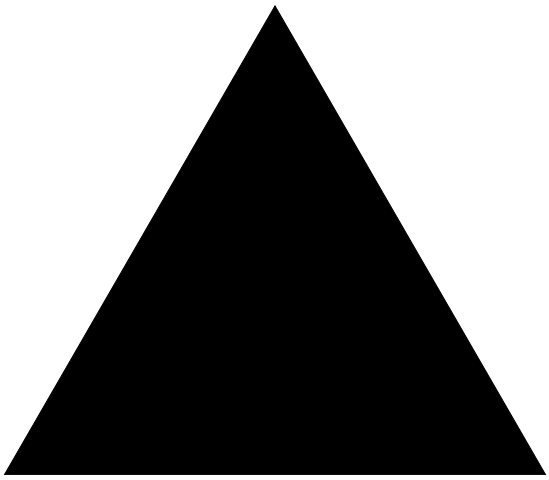


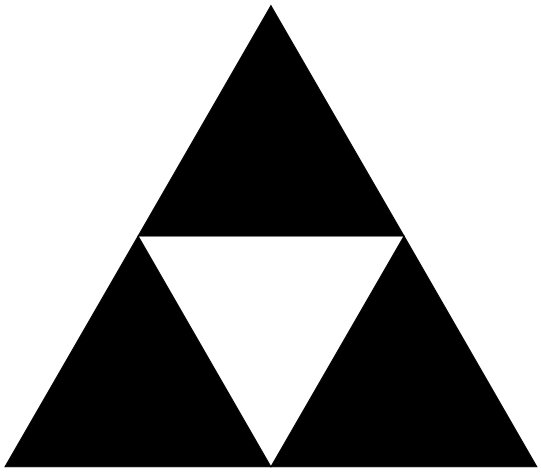
The blocks move in the direction in which the tab is pointing.

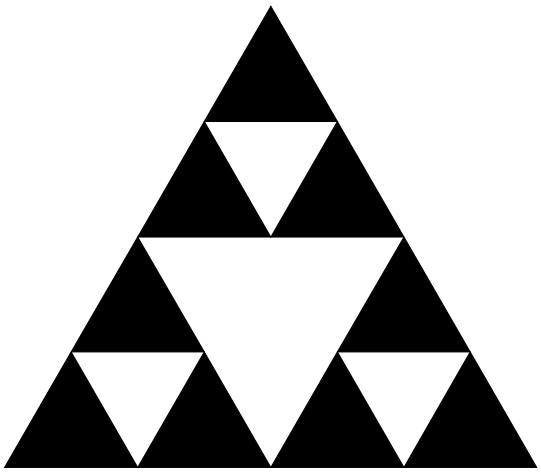
The Sierpinski Triangle

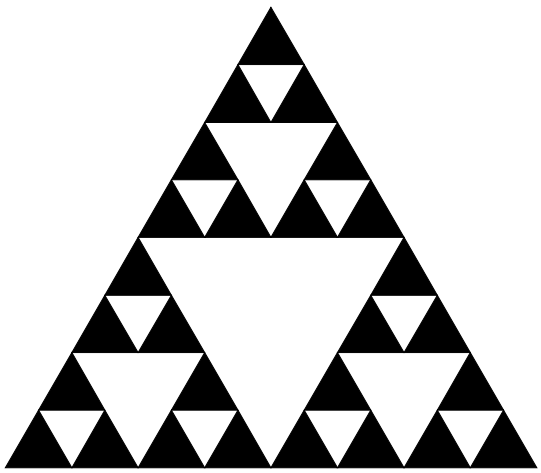


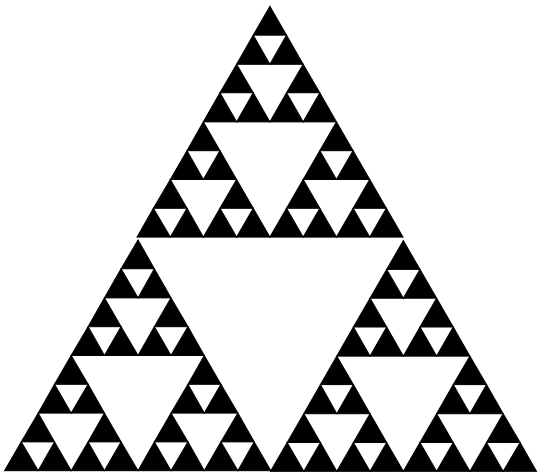
Generation of the Sierpinski Triangle











Can you give an iterative rule that generates this sequence of figures?

Counting Subsets

How many ways are there to select a committee from among 5 people?

Al	Bob	Cate	Deb	Edith

Fact: The "Standing Bosses" game with n players gives all possible 0-1 strings of length n.

(We'll prove this shortly. But for now, go ahead... give me a 0-1 string of length 5.)

Observation: We can "encode" a selection of a committee by using the code

1 = selected
0 = unselected

Bijection

There exists a natural bijection between 0-1 strings of length n and subsets of a set of size n (suitably ordered).

Al	Bob	Cate	Deb	Edith

$$
\begin{array}{ccc}
00000 & \leftrightarrow & \varnothing \\
00001 & \leftrightarrow & E \\
00010 & \leftrightarrow & D \\
00011 & \leftrightarrow & DE \\
\vdots & \vdots & \vdots \\
10101 & \leftrightarrow & ACE \\
\vdots & \vdots & \vdots \\
11111 & \leftrightarrow & ABCDE
\end{array}
$$

So the number of subsets of {A, B, C, D, E} is the same as the number of 5-bit strings

Proof that we Get All Strings

The standing bosses game is a (cheesy) coding of the binary increment. The game is exactly how we add "1" to an integer in binary.

```
 11010010110          11010010111
  +          1        +          1
-------------        -------------
```

We started with everyone sitting 0
We ended with everyone standing . . . $(2^{\#people} - 1)$

Those are respectively the smallest and largest binary values that a bit string can represent.

So all other bit strings must fall somewhere between them. Since our operation counted by ones, we must hit all other bit strings along the way from 0 to $2^{\#people} - 1$.

Another Proof that we Get All Strings

We can check by hand that for 3 players we get all strings.

```
000
001
010
011
100
101
110
111
```

Now suppose there are 4 players.
What does the leftmost player do?

- Watches a full game with 3 players
- Stands up
- Watches another full game with 3 players

We therefore get all 4-bit 0-1 strings.

```
0000
0001
0010
0011
0100
0101
0110
0111
1000
1001
1010
1011
1100
1101
1110
1111
```

How about 5?

The rest of the proof proceeds by induction on n, the length of the strings.

How Long do the Boss Games Last?

Assuming that every possible 0-1 string really does appear, the question becomes:

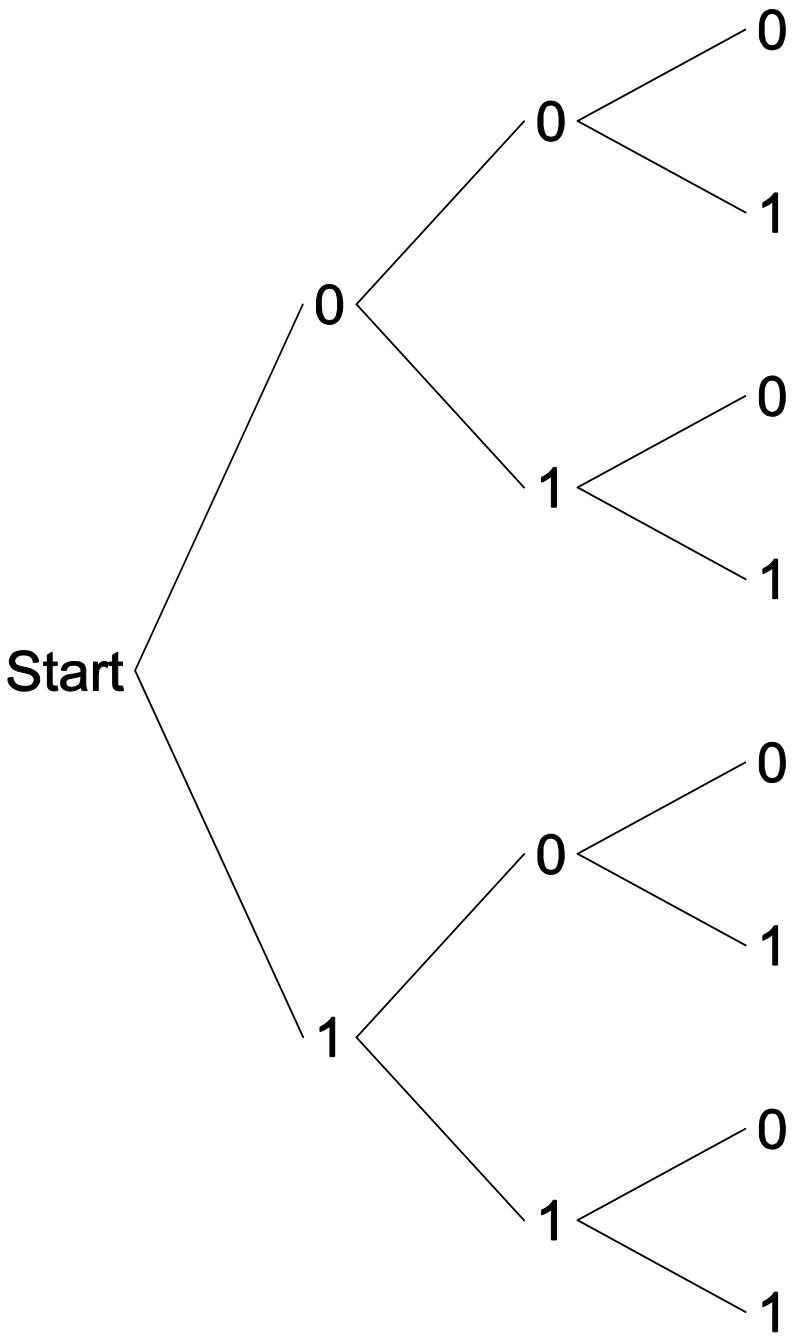
How many 0-1 strings are there?

(*Okay... I know you already know the answer.*)

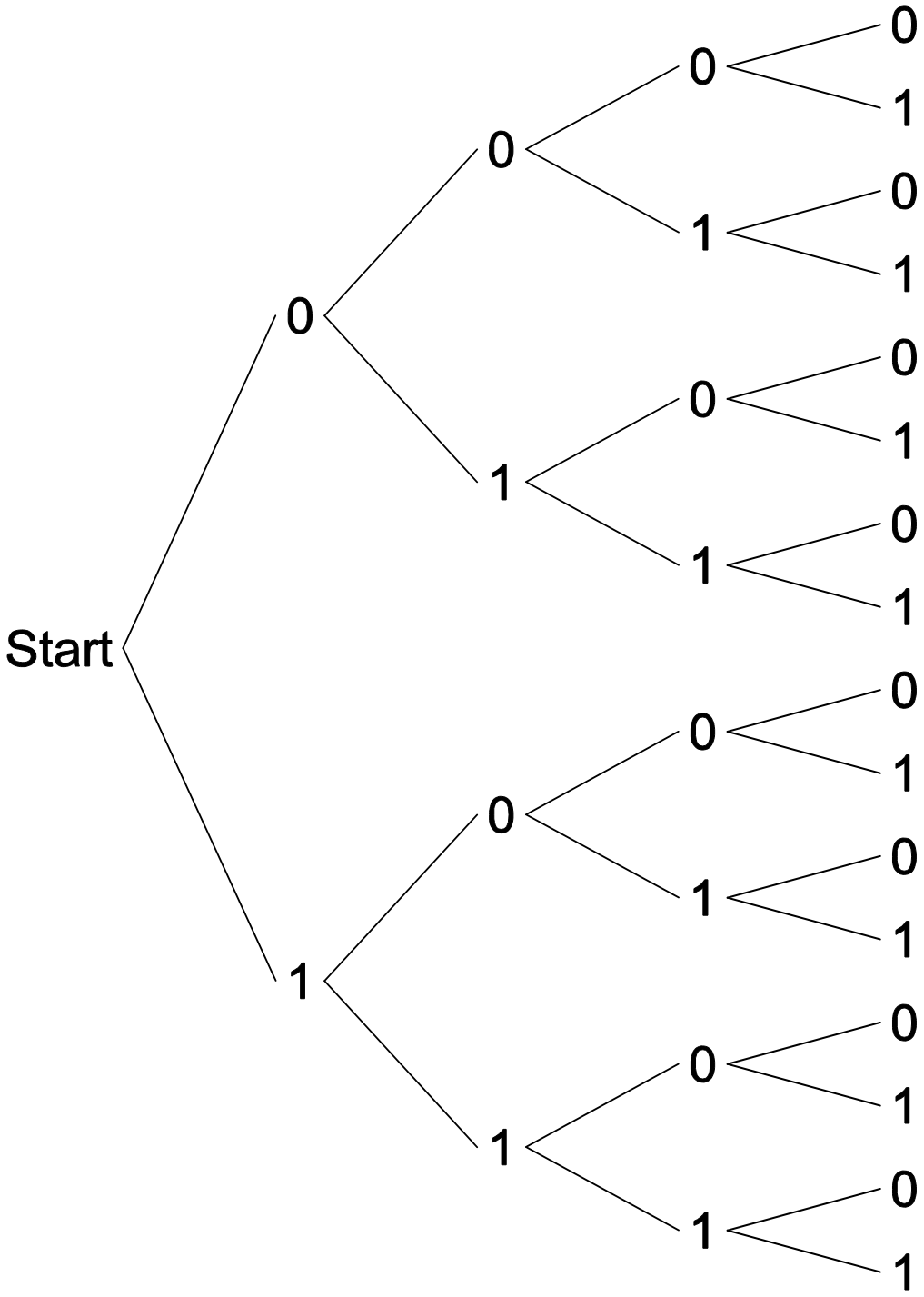
Let's answer this for strings of length 3 by means of a skill that way too many undergraduate students undervalue:

Make a Systematic List

Tree Diagram for 3 Players



Tree Diagram for 4 Players



In General?

We can find the number of binary strings of length n by doubling repeatedly:

$$2 \times 2 \times \ldots \times 2$$

where there are n 2s in that expression.
So the number of strings of n bits is 2^n.

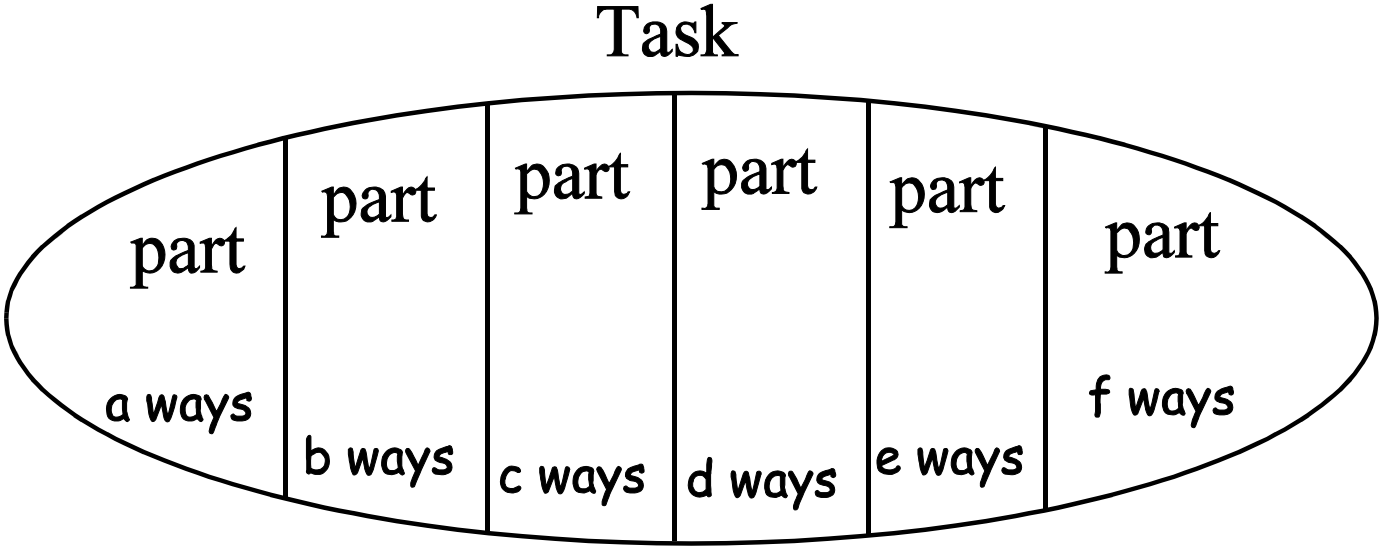
Another Way to Think of it

Picture n slots, each of which may be filled with either a 0 or 1.

___ ___ ___ ... ___

The Multiplication Rule

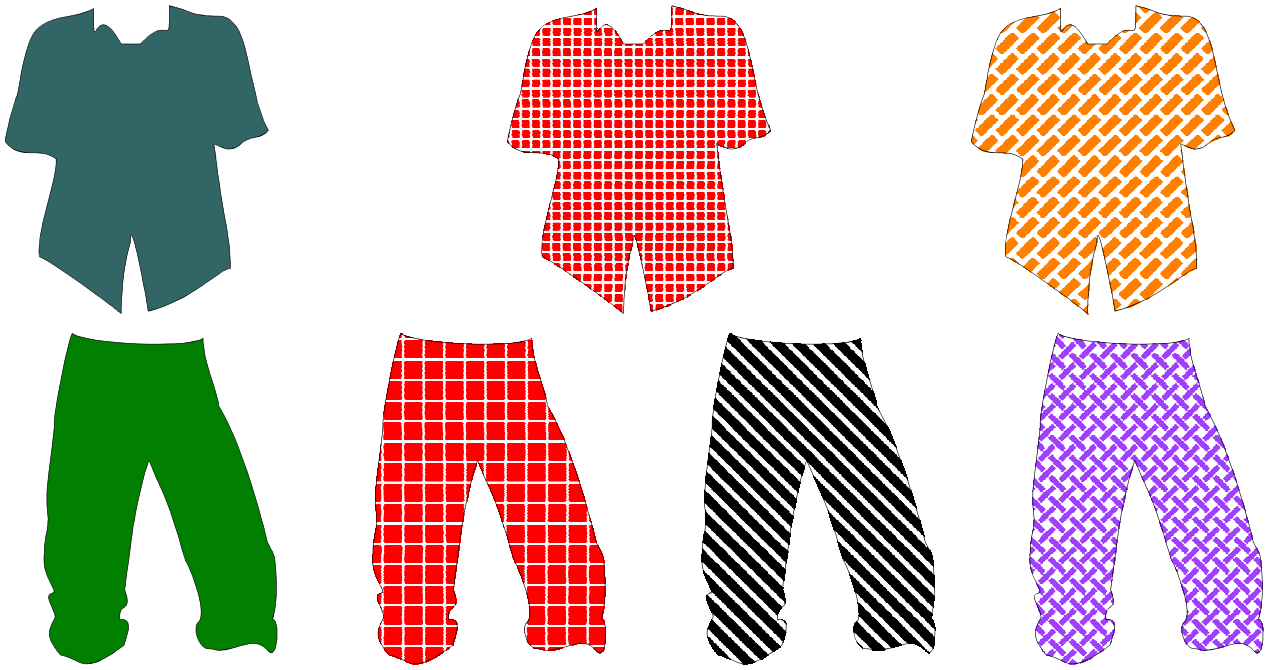
- If a **task** consists of several **parts**,
- And each part *must* be performed to complete the **task**
- Then the number of ways the **task** can be accomplished is the product of the number of ways to do each **part**.



The number of ways to complete this task is

$$a \times b \times c \times d \times e \times f$$

Shirts and Pants



Barry wishes to select an outfit for each day of a 10-day program. He has 3 shirts and 4 pairs of slacks available. How many outfits are possible? Will he have to repeat himself?

Code Words

The Dempsey Club assigns each of its members a 3-letter password which it uses for identification when a member “puts it on the tab.” The letters are selected from the word

AMBIDEXTROUSLY

How many 3-letter code "words" are possible?

The Pet Store

A pet store has for sale:

5 dogs
3 cats
4 fish

How many ways are there for little Suzy to select a pet to bring home?

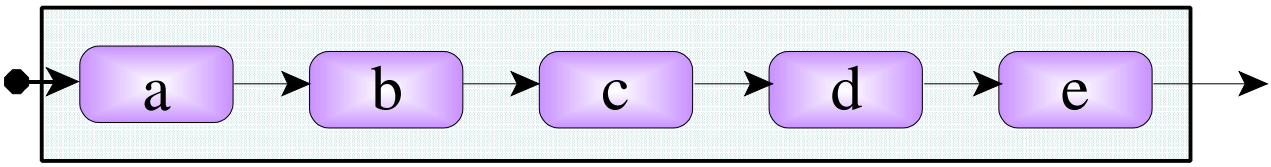
The multiplication rule does not apply. We do not have a task with many parts that all need to be performed.

Can you think of a question where the answer is $5 \times 3 \times 4$?

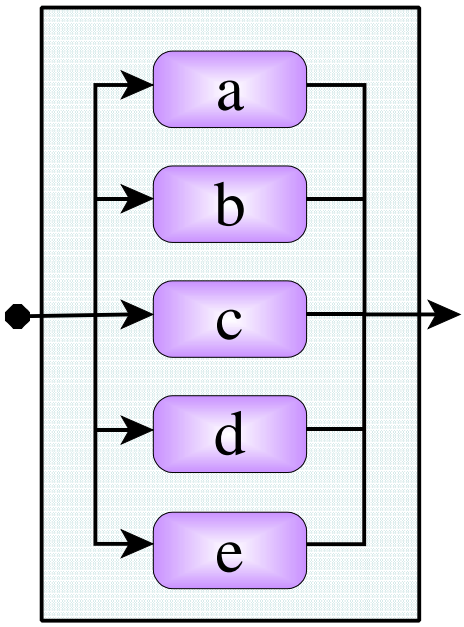
How many ways are there for little Suzy to select one of each type of animal?

Add or Multiply?

All parts must be performed — Multiply



Only one part must be performed — Add



Do whatever you can to help your students keep this straight!

Arrangements

In how many ways can the letters of the word

SPOT

be arranged?

OPST	POST	SOPT	TOPS
OPTS	POTS	SOTP	TOSP
OSPT	PSOT	SPOT	TPOS
OSTP	PSTO	SPTO	TPSO
OTPS	PTOS	STOP	TSOP
OTSP	PTSO	STPO	TSPO

Class Officers

Amy, Brenda, Clement and Darron have been selected as class officers. Now they need to be assigned to the positions of President, Vice President, Secretary and Treasurer.

In how many ways can this be done?

Amy	Brenda	Clement	Darron
_____	_____	_____	_____
Pres	VP	Secy	Treas

How Many Ways to...

- ...arrange your Ferrari, BMW and Mercedes in your 3-car garage?
- ...assign six beds to your six children?
- ...select the order in which you will surf, parasail, collect shells and sunbathe during your day at the beach?
- ...arrange your ten favorite novels on a bookshelf?
- ...select the order in which to remove seven illegal animals from the beach?
- ...decide which of your five employees will sweep, wash, cook, serve and bus, respectively?

Factorial

The number of ways to arrange n distinct objects is given by the expression

$$n \times (n - 1) \times (n - 2) \times \ldots \times 2 \times 1$$

This quantity arises often enough that it has been given its own notation:

$$n!$$

which is pronounced "n factorial."

Factorial

Here is a chart showing the first few factorials:

n	$n!$
0	
1	
2	
3	
4	
5	
6	
7	
8	
9	
10	

Some Counting Practice

1. A classroom has 12 boys and 14 girls, and the teacher wants to select one boy and one girl to be class representatives to the Spring Fling committee. In how many ways can the teacher make her selection?

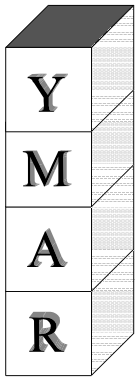
2. A multiple choice quiz has 5 questions, and 4 choices for each question. How many ways are there for a student to answer the questions on this quiz?

3. Dagwood is exercising great restraint in making a sandwich: He is going to use only one kind of meat, one kind of cheese, one topping and one kind of bread. The meat choices are *ham, salami, bologna, chicken roll* or *turkey*; the cheese he selects from *American, Swiss* or *provolone*; the topping he selects from *mustard, mayonnaise, butter, pickles* or *pepper*; the bread is either *rye* or *pumpernickel*. How many sandwiches are possible?

Some More Counting Practice

4. A preacher gave 33 sermons in 2000, 41 sermons in 2001 and 37 sermons in 2002. He wants to reuse a sermon from a previous year this Sunday. In how many ways can he do this?

5. How many different ways are there to stack the blocks shown to the right?



Some More Counting Practice

6. How many ways are there to build a stack of blocks from those shown in problem 5, where the stack can have any size?

7. In the game of Canaps a gambler rolls either two or three dice of different colors, and wins or loses depending on the outcome. How many outcomes are possible?

Some More Counting Practice

8. The host of a new game show is explaining the rules. The contestant can select either "star" or "year." If she selects "star" then she has to name the star of a certain sitcom, and the host has 20 sitcoms to select from. If she selects "year," then she has to name the year in which a certain movie won an academy award (the host has 8 to choose from) and then select either "sitcom," "soap" or "drama" and name the year in which a show from that category started airing (the host has 9, 17 and 5 respectively of each to choose from). In how many ways can this game be played?

Counting Anagrams

For us, an "anagram" of a word is any rearrangement of the letters, whether it makes sense in English or not.

For example, the following are anagrams of STAR:

ARTS
RATS
TARS

But so are:

STRA
SART
SATR
SRAT
SRTA
etc
⋮

We are interested in the question of how many anagrams there are of any given word

How Many Anagrams?

TO

TOE

TOES

TONES

TONERS

ATONERS

Handout #1 — What Patterns do you See?

The Game with 3 People	The Game with 4 People	The Game with 5 People
000	0000	00000
001	0001	00001
010	0010	00010
011	0011	00011
100	0100	00100
101	0101	00101
110	0110	00110
111	0111	00111
	1000	01000
	1001	01001
	1010	01010
	1011	01011
	1100	01100
	1101	01101
	1110	01110
	1111	01111
		10000
		10001
		10010
		10011
		10100
		10101
		10110
		10111
		11000
		11001
		11010
		11011
		11100
		11101
		11110
		11111

Handout #2 — Some Counting Problems

1. A classroom has 12 boys and 14 girls, and the teacher wants to select one boy and one girl to be class representatives to the Spring Fling committee. In how many ways can the teacher make her selection?

2. A multiple choice quiz has 5 questions, and 4 choices for each question. How many ways are there for a student to answer the questions on this quiz?

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6. How many ways are there to build a stack of blocks from those shown in problem 5, where the stack can have any size?

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8. The host of a new game show is explaining the rules. The contestant can select either "star" or "year." If she selects "star" then she has to name the star of a certain sitcom, and the host has 20 sitcoms to select from. If she selects "year," then she has to name the year in which a certain movie won an academy award (the host has 8 to choose from) and then select either "sitcom," "soap" or "drama" and name the year in which a show from that category started airing (the host has 9, 17 and 5 respectively of each to choose from). In how many ways can this game be played?

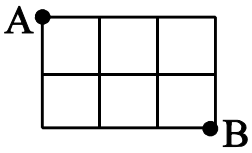
Exercises — Introduction to Counting

Warm-up Problems:

1. How many ways are there to arrange the letters of the word "LOVE"?

2. How many ways are there to make a 3-letter codeword from the letters of the word "AEGILOPS"? ("Aegilops" is the longest word in the English language with all letters in alphabetical order.)

3. How many ways are there to select three letters from the letters of the word "AEGILOPS"?



4. How many ways are there to walk from A to B on the grid to the right, without backtracking (that is, moving only south and east)?

Presentation Problems:

5. How many 10-digit numbers are there which don't use any digit more than once, where no odd digit ever follows an even digit, and where the "4" and the "5" are next to each other?

6. Pappy's Pizza offers seven different toppings for their pizzas. How many different ways are there to make a pizza, including "plain" and "the works?"

7. Consider the following variation on the "Standing Bosses" game:
   Line up some players, sitting in chairs, facing the class, then iterate:
   - The boss is the second-to-rightmost sitting player
   - The boss's job is to stand up, and to tell all standing players to his right to sit down

   The game is repeated for as long as possible
   For example, with 3 players, the game goes like: 000 → 010 → 100 → 110 and game over.
   a. The game with 3 players had 4 positions altogether. How many positions does the game with 4 players have altogether?
   b. How many positions do the games with 1, 2, and 5 players have?
   c. How many positions does the game with *n* players have?

8. Consider the following variation on the "Standing Bosses" game:
   Line up some players, sitting in chairs, facing the class, then iterate:
   - The boss is the rightmost sitting player who does not have a person standing to his *immediate* left
   - The boss's job is to stand up, and to tell all standing players to his right to sit down

   The game is repeated for as long as possible
   For example, with 3 players, the game goes like: 000 → 001 → 010 → 100 → 101 and game over.
   a. The game with 3 players had 5 positions altogether. How many positions does the game with 4 players have altogether?
   b. How many positions do the games with 1, 2, and 5 players have?
   c. How many positions does the game with 15 players have?

9. For each of the following words, make a tree diagram showing all of that word's anagrams:
   a. OX
   b. BOX
   c. BOO
   d. BOON
   e. NOON
   f. NOONE
   g. NOOSE

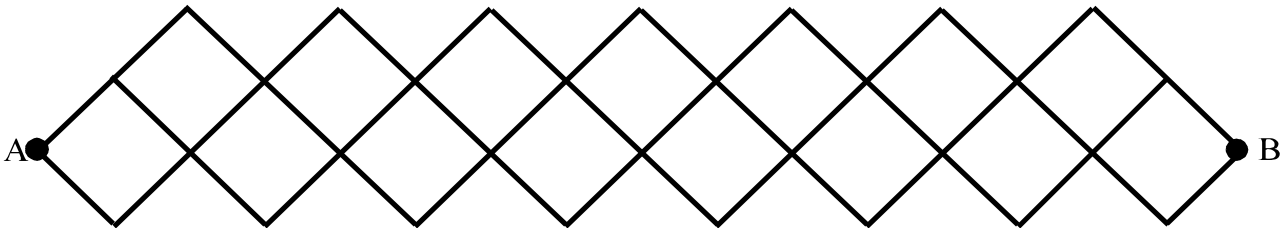
   For each of these words, give the number of anagrams.

10. In the game of Spiroletto, a single die is tossed and then a coin is tossed that number of times shown by the die. The player wins a dollar for each Heads that appears.
   a. How many game plays are possible, including the outcome for the die and coin tosses?
   b. What if it were an 8-sided die with the numbers 1-8 on its faces?
   c. What if it were an *n*-sided die with the numbers 1-*n* on its faces?

11. Bob is a subset lister, and Sam is an arrangement selecter. Bob is given a set of size *n* and has to make a list, in some order, of all subsets of that set. Sam is given a word with *n* distinct letters and has to give some selection of anagrams of that word.
   a. In how many ways can Bill and Sam do their jobs for $n = 0, 1, 2, 3$ and 4?
   b. For large values of *n*, whose job can be done in more ways? For this problem, you may wish to use logs, and consider Stirling's approximation: $n\,! \approx (n / e)^n \cdot \sqrt{(2\pi\, n)}$

Extension Problems:

12. How many ways are there to walk from A to B on the figure below, without backtracking?



13. Consider the following variation on the "Standing Bosses" game:
   Line up some players, sitting in chairs, facing the class, then iterate:
   - The boss is the person to the left of the leftmost sitting player, unless the leftmost sitting player is the leftmost player, in which case the boss is the rightmost player.
   - The boss's job is to stand up, and to tell all standing players to sit down

   The game is repeated for as long as possible. How many positions does the game with *n* players have?

14. Suppose we have some variant of the standing bosses game with the following properties:
   a. There is some unique start position
   b. A position consists of a row of players, fixed in number, each either sitting or standing
   c. There is a rule which, at each step, determines the next position from the current position alone, without using any randomness or any outside input.

   Prove that such a game on *n* players must either continue indefinitely, or stop within 2^n steps.

Discrete Math Resource Book
on
Introduction to Counting

Workshop Outline — Introduction to Counting

I. Opener — Standing Bosses
   - A. Start with 8 players
     1. See what happens
     2. Taking too long
   - B. Try 3 players
     1. Count steps
   - C. Try 4 players
     1. Count steps
   - D. Discussion of some patterns
   - E. There is a bijection between the set of 0-1 strings of length n and the subsets of a set with n elements
   - F. Two proofs that the standing boss sequence encodes all 0-1 strings of length n
     1. They encode the numbers from 0 to $2^n - 1$, each exactly once, in binary
     2. By induction
     3. Since there are 2^n 0-1 strings of length n, we know there are that many steps in the standing bosses game, and that many subsets of a set of size n

II. Introduction to Counting
   - A. First of all, there may be many different degrees of prior skill present. Please be understanding of this and adjust your participation accordingly
   - B. Multiplication rule
     1. Counting all the binary strings of length n
        - a. Tree Diagram as motivation
        - b. Using the multiplication rule
     2. Practice with other examples
        - a. Distribute handout
   - C. Addition rule
     1. Contrast with multiplication rule
     2. Distribute handout with mixed examples of when to add and/or multiply
   - D. Number of arrangements of n distinct objects
     1. Factorial