Ordered pairs and *n*-tuples

- The *ordered pair* (a, b) is a collection that has a as its first element and b as its second element
  - Note how this is different from a set; in sets the elements are not ordered
  - The set $\{1, 2\}$ is the same as the set $\{2, 1\}$, but the ordered pairs $(1, 2) \neq (2, 1)$
- The *ordered triple* (a, b, c) is a collection with a as its first element, b as its second element and c as its third element.
- The *ordered n-tuple* $(a_1, ..., a_n)$ is the collection which has a_i as its ith element, for $1 \leq i \leq n$

Cartesian Products

- The Cartesian product gives us a way to "multiply" sets together
- If A, B, ..., Q are sets, then the Cartesian product $A \times B \times \cdots \times Q$ is the set $\{(a, b, ..., q) \mid a \in A, ..., q \in Q\}$
- Note that the Cartesian product of k sets is a set of ordered k-tuples
- What is {x, y}×{p, q}
  - {(x, p), (x, q), (y, p), (y, q)}
- What is {1, 2, 3}×{a, b}×{5, 6}?
  - {(1, a, 5), (1, a, 6), (1, b, 5), (1, b, 6), (2, a, 5), (2, a, 6), (2, b, 5), (2, b, 6), (3, a, 5), (3, a, 6), (3, b, 5), (3, b, 6)}

A Familiar Cartesian Product

- You’ve probably heard the term “Cartesian coordinates”
- It usually refers to the (x, y)-coordinate system we use when graphing in the plane
- or (x, y, z)-coordinates when graphing in space
- If we let $\mathbb{R}$ denote all real numbers (the number line), then the plane is $\mathbb{R}\times\mathbb{R} = \mathbb{R}^2$
- and space is $\mathbb{R}\times\mathbb{R}\times\mathbb{R} = \mathbb{R}^3$

Union of Sets

- The *union* of A and B is the set that contains all elements found in either A or B
- $A \cup B = \{x \mid x \in A \vee x \in B\}$
- Eg: $A = \{1, 2, 3\}$, $B = \{3, 4\}$, $C = \{1, 4, 7\}$.
  - What is $A \cup B$?
  - What is $A \cup C$?
  - What is $A \cup B \cup C$?
- What is $A \times B \cup C$
  - It depends on where you put the parentheses!
  - $(A \times B) \cup C = \{(1, 3), (2, 3), (3, 3), (1, 4), (2, 4), (3, 4), 1, 4, 7\}$
  - $A \times (B \cup C) = \{(1, 1), (1, 3), (1, 4), (1, 7), (2, 1), (2, 3), (2, 4), (2, 7), (3, 1), (3, 3), (3, 4), (3, 7)\}$

Intersection of Sets

- The *intersection* of A and B is the set that contains all elements found in both A and B
- $A \cap B = \{x \mid x \in A \wedge x \in B\}$
- Eg: $A = \{1, 2, 3\}$, $B = \{3, 4\}$, $C = \{1, 4, 7\}$.
  - What is $A \cap B$?
  - What is $A \cap C$?
  - What is $A \cap B \cap C$?
- What is $A \times B \cap C$
  - $(A \times B) \cap C = \varnothing$
  - $A \times (B \cap C) = \{(1, 4), (2, 4), (3, 4)\}$

Set Difference

- The *difference* of A and B is the set that contains all elements found in A, but not B
- $A - B = \{x \mid x \in A \wedge x \notin B\}$
- Eg: $A = \{1, 2, 3\}$, $B = \{3, 4\}$, $C = \{1, 4, 7\}$.
  - What is $A-B$?
  - What is $A-C$?
  - What is $A-B-C$?
    - Here it depends on where you put the parentheses!
    - $(A-B)-C = \{2\}$
    - $A-(B-C) = \{1, 2\}$
  - What is $A\times B-C$
    - $(A\times B)\cap C = \varnothing$
    - $A\times(B\cap C) = \{(1, 4), (2, 4), (3, 4)\}$

Complement of a Set

- The complement of a set is the set of all elements that are not in that set.
  - Does this make sense?
  - What's missing?
- Before defining the complement of a set, we must have the notion of a *universal set*
  - The universal set is the set of all elements under consideration. This must be made clear.
- This is analogous to the *universe of discourse* for predicates
- All together: *I will not write "compliment" of a set*

Complement of a Set

- Suppose U is the universal set. Then for a set A,

$$\overline{A} = \{x \in U \mid x \notin A\}$$

- Thus $U - A$ is another way of writing the complement
- Suppose U is the set of all integers
  - $A = \{x \mid x \text{ is an even integer}\}$
    - complement = set of odd integers
  - $B = \{x \mid x \text{ is a multiple of } 3\}$
    - complement = set of integers which leave a remainder of 1 or 2 when divided by 3

Set Identities

- Several set identities are give on page 49:

$A \cup \varnothing = A$	$A \cap U = A$
$A \cup U = U$	$A \cap \varnothing = \varnothing$
$A \cup A = A$	$A \cap A = A$
$\ldots$	
$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$	$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

- How would one prove such identities?
- Three ways:
  - Membership Tables
  - Use of simpler identities
  - In general, to show two sets are equal, it is enough to show that each set is a subset of the other set

Membership Tables

- For each simple set in the expression, make a column that will indicate whether an element is (1) or is not (0) in that set
- For more complicated columns, combine the simpler columns
- For example, let us prove one of DeMorgan's laws: *The complement of the union is the intersection of the complements*

Membership Tables

A	B	$A \cup B$	$\overline{A \cup B}$	$\overline{A}$	$\overline{B}$	$\overline{A} \cap \overline{B}$
1	1	1	0	0	0	0
1	0	1	0	0	1	0
0	1	1	0	1	0	0
0	0	0	1	1	1	1

Prove $(A-B) \cap C \subseteq C - B$

- This time, what would we require of the membership table?
  - That every "1" on the left-hand side has a corresponding "1" on the right-hand side.
  - But there could be more "1s" on the RHS than the LHS

A	B	C	A−B	(A−B)∩C	C − B
1	1	1	0	0	0
1	1	0	0	0	0
1	0	1	1	1	1
1	0	0	1	0	0
0	1	1	0	0	0
0	1	0	0	0	0
0	0	1	0	0	1
0	0	0	0	0	0

Every "1" in this column

Has a corresponding "1" in that column, proving the desired subset relationship.