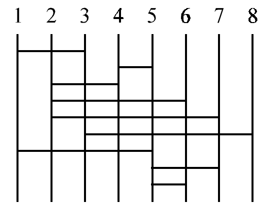
Some Sample Problems in Advance of the First Exam.



Web Diagram A

1. What permutation is generated by web diagram A:

2. Is it possible to add one line at the bottom of the diagram so that player 5 has to take notes on the last day?

3. How about at the top of the diagram?

4. How about as the 5th line down, beneath the line connecting 2 and 6?

5. Show how, given any web diagram, any height, any player and any desired position, it is possible to add a single line at the given height so that the given player ends up in the desired position.

6. What is an inversion in a permutation?

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Permutation B

7. How many inversions does permutation B have?

8. What is meant by the parity of a permutation?

9. What is the parity of permutation B?

10. If I switch the 6 and the 8 in permutation B to obtain a new permutation, how many inversions will the resulting permutation have?

11. In general, what happens to the number of inversions when one swaps a single pair?

12. What is the parity of the permutation generated by web diagram A? How can you tell simply by looking at the web diagram?

13. Given a permutation of *n* elements, what is the largest number of lines that could be needed in a web diagram which generates that permutation?

14. I have a web diagram with 22 lines generating a permutation on 10 elements. Prove that there is also a web diagram generating this permutation that has no more than 8 lines.

15. An “odd cycle” is a permutation of an odd number of elements consisting of a single cycle, and an “even cycle” is a permutation of an even number of elements consisting of a single cycle. For example, (23451) is an odd cycle. What is the parity of an odd cycle? What is the parity of an even cycle?

16. Arrangements of the 15 tiles on the Fifteen Puzzle can be thought of as permutations of the numbers 1-15, with the blank tile always on the lower right. What sorts of permutations are solvable?

17. How would you prove your answer to problem 16?

18. The game board to the right shows three rows of numbers. To win this game, you must return all three rows to the home position "1 2 3 4." The moves are made as follows: First, swap any pair you wish in row 1, then any pair you wish in row 2, then any pair you wish in row 3, then any pair you wish in row 1, then row 2, then row 3, then row 1, etc... You must progress through the rows in the order indicated, and you must make exactly one swap each time you get to a row. Prove that even though this game is a load of fun, it cannot be solved.

4	2	3	1
4	3	2	1
1	3	2	4

Game C

19. If we build a web diagram to generate permutation B, but our lines may connect only adjacent vertical lines, what is the least number of lines we would need?

20. If I asked you to prove your answer to the above question was correct, what are the *two* things you would need to prove?

21. Let π be a permutation, let $t(\pi)$ denote the number of inversions in π and $l'(\pi)$ denote the least number of lines needed in a web diagram to generate π using lines that connect adjacent vertical lines. Prove that $l'(\pi) = t(\pi)$.

22. What is the cycle notation for the permutation generated by web diagram A?

23. What is the cycle notation for permutation B?

24. What is the cycle notation for each of the three permutation given in game C?

25. Draw permutation digraphs for each of the permutations asked for in the last 3 problems.

26. Suppose a permutation ladder generates a permutation which consists of three cycles, of sizes 5, 6 and 7 respectively. What are all the possible cycle structures of the permutation that results when a line is added to the bottom of the permutation ladder?

27. Can the 15-puzzle shown to the right be solved:

28. Suppose I swap 1 and 14 in the puzzle shown to the right. Can the resulting puzzle be solved?

29. How many permutations of the numbers 1, 2, 3, 4 are there? How many of those permutations are even? How many are odd?

1	**3**	**6**	**10**
2	**5**	**9**	**13**
4	**8**	**12**	**15**
7	**11**	**14**	

Homework 2 — Adjacent Transpositions

Due on Tuesday, February 10

Here are some definitions:

Permutation Ladder:	A diagram of horizontal and vertical lines which generates a permutation
Inversion:	A pair of elements which is out of order compared to the "home" position

Here is some notation:

Let π be a permutation of the elements $\{1, 2, ..., n\}$

$t(\pi)$	The number of inversions in π
parity(π)	The parity of $t(\pi)$, that is, whether $t(\pi)$ is odd or even
$l(\pi)$	The least number of rungs needed in a permutation ladder to obtain π

Question1:

Prove that $t(\pi)$ and $l(\pi)$ always have the same parity

Question 2:

Suppose the rungs in a permutation ladder for π were allowed to connect only adjacent pairs of vertical lines. Find a formula, in terms of the notation given above, for the minimum number of rungs needed to generate a given permutation π.

Question 3:

Prove that fewer rungs will not suffice

Question 4:

Prove that you can always succeed with no more than that many rungs

Now solve problems 1, 7, 9, 14, 18, 19, 23, 26 and 27 from the practice problems sheet, and turn them in with these problems.