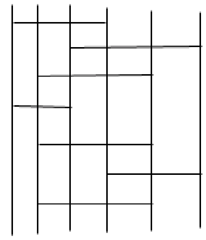
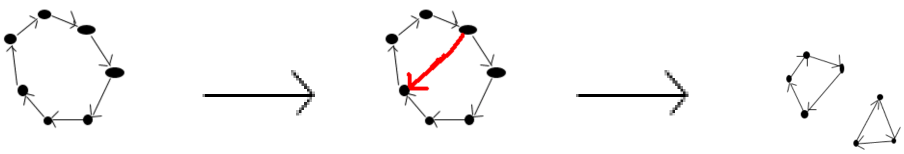
Discrete Mathematics — Class 5 — Thursday, January 27, 2004

If this ladder generates some permutations with **c** cycles and you add a new rung at the bottom then the new permutation will have _____ how many cycles?

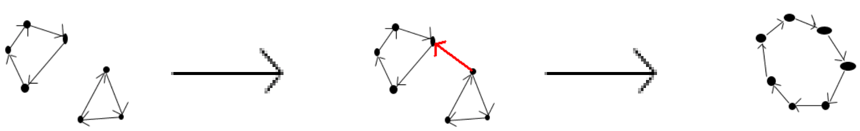


It will have either **c + 1** cycles or **c – 1** cycles.

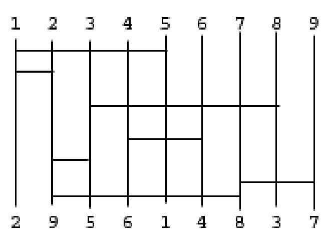
- **c + 1**: it will have c + 1 cycles if the new rung connects elements within an existing cycle. It will split the "cycle structure" into two separate pieces.



- **c – 1**: it will have c – 1 cycles if the new rung connects elements in separate existing cycles. It will join the two cycles into one big cycle.



Ex. Given the following permutation, 2 9 5 6 1 4 8 3 7, build a permutation ladder with the least number of rungs. What is the least number of rungs you need to build the ladder?



You can solve this problem by using the following equation where *n* = number of elements in the permutation (9 in above example) and *c* = number of cycles (2 in above example).

$$n - c = \text{minimum number of rungs}$$

Theorem: The minimum number of rungs needed in a permutation ladder to generate a permutation on ***n*** elements with ***c*** cycles is ***n* – *c***.

Example of how to use theorem: Given the following permutation, what is the least number of rungs needed?

2 9 5 6 1 4 8 3 7

Solve: 1) Determine cycle structure: (1 5 3 8 7 9 2) (4 6) ***c* = 2**
2) Determine number of elements (count number of elements): ***n* = 9**

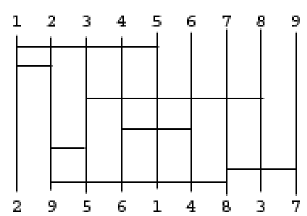
n – *c* = minimum number of rungs → 9 – 2 = 7 → 7 is minimum number of rungs needed.

Is it possible to draw a permutation ladder to make the permutation above with
- 6 rungs? NO – needs a minimum of 7 rungs!
- 5 rungs? NO – needs a minimum of 7 rungs!
- 3 rungs? NO – needs a minimum of 7 rungs!
- 8 rungs? YES – this works because it is greater than the minimum number of rungs needed – 7.

Proof Of Theorem

Note: There are really two (2) things to solve:
1) You **CAN NOT** do it with fewer than ***n* – *c*** rungs. (Impossible to do it with less than ***n* – *c***).
2) You **CAN** do it with ***n* – *c***.

Proof of 1:

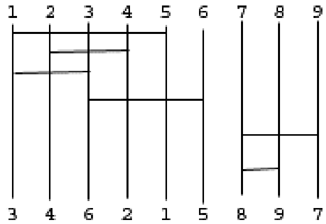


- Home position has 9 cycles (each element is its own cycle).
- In final permutation there are 2 cycles: (1 5 3 8 7 9 2) (4 6)
- Every time you add a rung you create a new permutation until you get to the final permutation. And each rung changes the number of cycles by exactly 1.

In general, to go from ***n*** cycles to ***c*** cycles, where each rung can decrease the number of cycles at most one (1) at a time, we will need greater than or equal to ***n* – *c*** rungs

*** This shows that the best you can hope for is ***n* – *c*** rungs.

Proof of 2:



We need to find a way to do it with no more than ***n – c*** rungs.

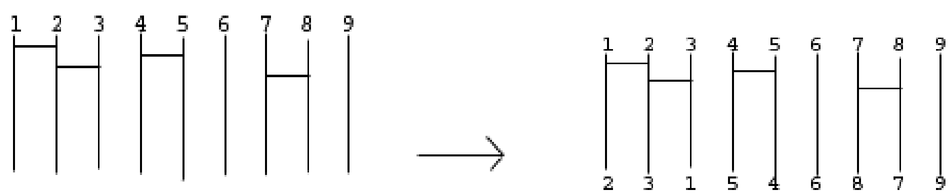
We determined through group work that there are two techniques to show that the minimum number of rungs needed is actually ***n – c***.

Solution (technique) 1: Start at top of elements and work your way down one-by-one. This will always give you the best answer (minimum rungs required). Unfortunately, this technique does not lead us quickly to a proof that $n - c$ rungs will always suffice.

Solution (technique) 2: Start by separating the cycle structures and then do one cycle tructure at a time (fix one cycle at a time).

Ex.	2 9 5 6 1 4 8 3 7	3 4 6 2 1 5 8 9 7
Cycle structure:	(1 5 3 8 7 9 2) (4 6)	(1 5 6 3) (2 4) (7 9 8)
At most $n - 1$ Rungs per cycle	(7 – 1) +(2 – 1)	(4 – 1) +(2 – 1) +(3 – 1)
	=(9 – 2)	= (9 – 3)
	= ***n – c***	= ***n – c***

Ex. Start with the following permutation ladder and find the final permutation:



Cycle Structures: (1 3 2) (4 5) (6) (7 8) (9) $n = 9$
(3-1) (2-1) (1-1) (2-1) (1-1) $\boldsymbol{c} = 5$

= 9 – 5 = 4
= ***n – c***

<u>Proof</u>: To do it with ***n – c*** rungs, simply fix the cycles in the permutation one at a time.

Let $Size_1$, $Size_2$, $Size_3$, ….. , $Size_C$ stand for the sizes of the cycles.

- We can fix the first cycle with ($Size_1$ – 1) rungs.
- We can fix the second cycle with ($Size_2$ – 1) rungs.
- We can fix the third cycle with ($Size_3$ – 1) rungs.
- We can fix the **c** cycle with ($Size_C$ – 1) rungs.

Altogether you will need: ($Size_1$ – 1) + ($Size_2$ – 1) + ($Size_3$ – 1) +…+ ($Size_C$ – 1)

$$= (Size_1 + Size_2 + Size_3 + \ldots + Size_C) - (1 + 1 + 1 + \ldots + 1)$$

$$= \boldsymbol{n - c}$$

<u>Summary</u>

- Part 1: Number of rungs needed to generate permutation is at least ***n – c*** rungs.
- Part 2: ***n – c*** is sufficient for solving permutations and you will never need more.