# Discrete Mathematics — Class 4 — Thursday, January 22, 2004

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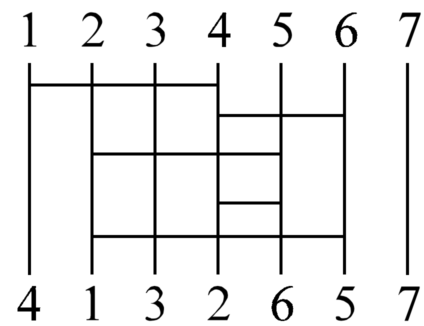
The professor (a substitute today, professor Pravica from the Math department) put a permutation ladder on the board, as shown to the right.

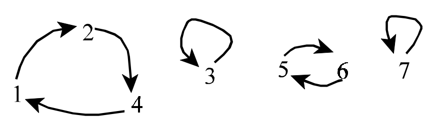



As noted in previous classes, every permutation is just a bunch of cycles (or maybe just one cycle). To find the "cycle structure" of this permutation, let's see where each element goes.

1 goes to 2's position
2 goes to 4's position
3 goes to 3's position
4 goes to 1's position
5 goes to 6's position
6 goes to 5's position
7 goes to 7's position.

Note that we can read this information off of the permutation ladder by "reading up." For example, to see where "2" goes, we look in the permutation row at the bottom of the ladder, and read up to see in which position 2 ended; in this case, 4's position.

The information above can be captured in a permutation digraph, which would look like this:




But a much more convenient way to capture this information is by using "cycle structure" notation, in which each cycle is enclosed in parentheses, and inside each set of parentheses, each element goes to the element to its right, with the last element in the set going back to the first element. This is called *cycle notation* for a permutation. The cycle structure for this permutation is (1 2 4)(3)(5 6)(7).

Note that every element in the permutation must be shown in the cycle structure, even the fixed points which lie alone in their parentheses. Also note that there are usually many ways to write cycle notation for a given permutation. For example, the above cycle structure could have been written:
(1 2 4)(7)(6 5)(3) or (4 1 2)(3)(7)(6 5), or any of 141 other ways.

What happens if we add a rung at the bottom of the permutation ladder? If a rung is added from 2 to 7 then the cycle structure becomes (1 2 7 4) (3) (6 5). The number of cycles decreased by 1 in this case.

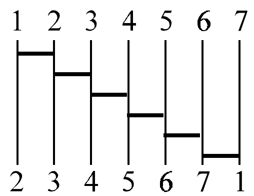
If we add a rung from 2 to 4, the cycle structure would become: (1 4)(2)(3)(5 6)(7). In this case, the number of rungs increased by 1.

It turns out that adding a rung to the bottom of a permutation ladder will always change the number of cycles in the cycle structure of the permutation by 1. It will:

Increase the number of cycles by 1 if the line connects two elements in the same cycle in the original permutation, because that cycle gets split into two cycles

Decrease the number of cycles by 1 if the line connects two elements in different cycles in the original permutation, because those two cycles will be merged into a single cycle.

If you create a ladder such as:




Then there is one cycle (1 7 6 5 4 3 2). This is consistent with the bulleted points above, because we started with home position (which has 7 elements, and hence 7 cycles) and ended with a permutation with 1 cycle, after 6 rungs were added.

We had already proved in class that any permutation on *n* elements will take at most $n - 1$ rungs to generate. In fact, for a cycle, this is the minimum number of rungs that will suffice. This leads to a general formula for the minimum number of rungs that will suffice to generate a permutation on *n* elements having *c* cycles:

**General result:** If you have *c* cycles for *n* numbers you will need $n - c$ rungs
This is the minimum amount you need

**Outline of proof:**
Proof has 2 parts: First observe that you can't do it with fewer rungs, because the number of cycles is *n* when there are no rungs, and each rung changes the number of cycles by 1. The goal is a permutation with *c* cycles, and each rung can change the number of cycles by at most 1. Thus we need at least $n - c$ rungs.

Second, observe that you *can* find a permutation ladder with $n - c$ rungs which generates the permutation. For simply fix each cycle in the permutation individually:
This will require at most (*size* – 1) rungs for a cycle of with *size* elements. If we add these quantities together for the cycles in the permutation, we get $n - \underline{c} = \sum$(size of each cycle -1), where the sum is over all the cycles.

More details in the next notes!