Discrete Math — Day 2 — 1/15/03

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1 15 Puzzle

The 15 puzzle is composed of 15 tiles in a 4 by 4 grid, with one blank space. The goal is take a scrambled board and then get them back into the correct order. Two examples can be seen in Figure 1 and Figure 2. The tiles in a 15 puzzle can only be moved by sliding a tile into the blank square, essentially swapping the two.

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	

Figure 1: 15 Puzzle in home (identity) form

1	6	2	4
9	5	3	7
10	14	12	8
13		11	15

Figure 2: 15 Puzzle in scrambled form

2 8 Puzzle

1	2	3
4	5	6
7	8	

Figure 3: 8 Puzzle

The 8 puzzle is a simplified version of the 15 puzzle, as seen in Figure 3 that was used for in-class demonstrations. Four 8 Puzzles were draw on the board (using cards for the tiles) and 4 groups of two were picked. For the first group, 1 pair of tiles in the 8 puzzle were swapped. For the second group, 2 pairs, and so on. The groups then were tasked to get their puzzles back to the home position.

Groups 2 and 4 were able to finish theirs fairly quickly, but groups 1 and 3 were unable to finish at all. As it turns out, puzzles with an odd number of switches are not solvable, while all puzzles with an even number of swaps are solvable with some number of swaps.

3 Parity

Every permutation has a **parity**, which can be either even or odd. A demonstration was performed with a the numbers 1-7 in a random order, representing a permutation. A number of cards were passed out to the class, each with two numbers (between 1 and 7) on it, "3 6" or "1 2" for instance. The two numbers were in order. The demonstration tracked what pairs of numbers were out of order by having the person with the corresponding card move to the front of the room. For instance, if the permutation was "1 2 4 3 6 5 7" given, those with the cards "3 4" and "5 6" would stand up. Various swaps were tried. The following pattern emerged: If two adjacent cards are swapped, one person will change position. If two cards with one in between are swapped, three people. If two cards with two in between are swapped, five people. If two cards with three in between are swapped, 7 people, and so on.

The number of people out of order (even or odd) is one way of determining the parity of the permutation. One thing that is important to note is that the number of people who move is always odd. This means that every swap will change the parity of the permutation.

4 Definition Of Parity

The parity of a whole number is if it is even or odd. The parity of a permutation, let's call it π, is the number of pairs that are out of order in π, relative to the home position.

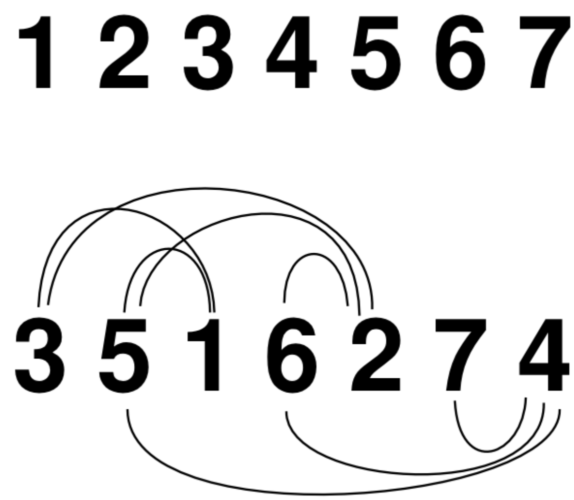

Figure 4: Finding the Pairs

5 Ways of finding the parity

Figure 4 is one way, a visual diagram of the out-of-order pairs. Consider the pairs of numbers, and draw lines between the ones that are out of order. Another term for a pair that is out of order is an "*inversion*".

An **inversion** in a permutation is a pair that is out of order relative to home position.

Another way to determine the parity is simply to solve the permutation, and count how many swaps that are required. The actual number is swaps isn't really important, just if it is even or odd. An even permutation will always require an even number of swaps, and an of permutation will always require an odd number of swaps. This is because the home position is even, and every swap switches the parity.

There is yet another graphical method for solving that also works, as shown in Figure 5

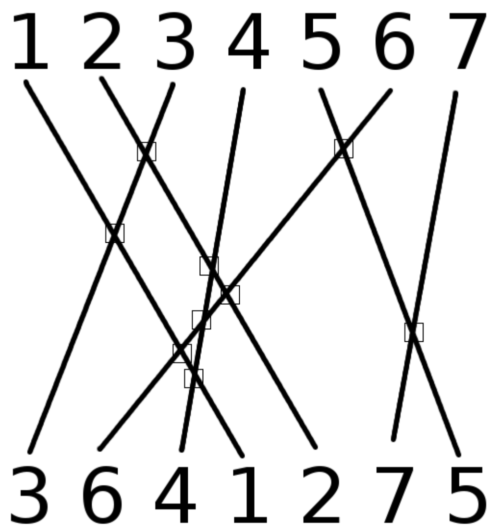

Figure 5: Line Crossing Method

Write out both the home position and the permutation, one above the other. Draw a straight line from each number to itself. The lines don't need to be perfectly straight, but it is important that only two lines cross at any given point. Count the number of crossings. This is the number of pairs that are out of order, from which the parity can be easily derived, as discussed earlier.