Complete Homework Solutions for Chapters 0, 1 and 2

Chapter 0

Problem 2: This one is done by building a truth table with columns for both $p \wedge (q \wedge r)$ and $(p \wedge q) \wedge r$.

Problem 4: Again, this is done with truth tables:

p	T	F	$T \wedge p$	$F \vee p$
t	t	f	t	t
f	t	f	f	f

Notice that the 4th column is the same as the 1st, and the 5th column is the same as the 1st.

Problem 6: If n is odd, then $n = 2k + 1$ for some integer k. This give $n^2 = (2k+1)^2 = 4k^2 + 4k + 1$. To show this is odd, we rewrite it as $2(2k^2 + 2k) + 1$, that is, as two times an integer plus one, showing

that n^2 is odd.

Problem 8: We will prove this by contradiction. Suppose that $\sqrt(3)$ is rational, that is, $\sqrt(3) = a/b$ for some integers a and b. Further, since every fraction can be expressed in lowest terms, we will assume that the fraction a/b is in lowest terms, that is, a and b have no common factor.

Squaring both sides we get $3 = a^2/b^2$, or $a^2 = 3b^2$. Now we will show that a must be a multiple of 3.

Claim: If a^2 is a multiple of 3, then a must be a multiple of 3.

Proof of Claim: Let r be the remainder obtained when we divide a by 3. Suppose for a contradiction that a is not a multiple of 3, which would mean that r is 1 or 2. Then $a = 3k + r$ for some integer k, and $a^2 = 9k^2 + 6kr + r^2 = 3(3k^2 + 2k) + r^2$. If $r = 1$, then $r^2 = 1$, and if $r = 2$, then $r^2 = 4$. In either case, it is not a multiple of 3, contradicting the fact that a^2 is a multiple of 3. □

Since a is a multiple of 3, we can write $a = 3l$ for some integer l. Substituting into $a^2 = 3b^2$, we get $(3l)^2 = 3b^3$, which simplifies to $3l^2 = b^2$. This implies that b^2 is a multiple of 3. But then by the claim above, b must also be a multiple of 3. This gives us a contradiction, since if a and b are both multiples of 3, then the fraction a/b was not in lowest terms. ■

Problem 10: We wish to prove that n^3 is odd if and only if (that is what "iff" means) n is odd. This requires two proofs: A proof that if n^3 is odd then n is odd, and a proof that if n is odd then n^3 is odd.

Such things are often written using the left-right arrow, and then arrows are used to indicate the implication that is about to be proved. I'll use this notation: **Theorem:** n^3 is odd $\leftrightarrow$ n is odd

Proof: ($\rightarrow$) We will prove the contrapositive. That is, we will prove that if n is not odd, then n^3 is not odd. Suppose that n is not odd. Then n is even, and we can write $n = 2k$ for some integer k. But then $n^3 = 8k^3$, which is even, so that n^3 is not odd.

($\leftarrow$) Now suppose that n is odd. Then we can write $n = 2l + 1$ for some integer l, which gives that $n^3 = (2l+1)^3 = 8l^3 + 12l^2 + 6l + 1 = 2(4l^3 + 6l^2 + 3l) + 1$ which is odd. ■

Problem 12: We will prove by contradiction that the set of integers is infinite. Suppose that the set of integers is finite. Then there must be some largest element of the set of integers. Let's call this element N. The quantity $N + 1$ is clearly also an integer, but it is larger than N, contradicting that N was the largest. Thus the set of integers cannot have a largest element, and therefore cannot be finite. ■

Problem 14: Let w, x, y and z be these integers, with $w < x < y < z$, having an average of 20. Let us assume (for a contradiction) that all numbers are 21 or less, so that $z \leq 21, y \leq 20, x \leq 19$ and $w \leq 18$. But then $w + x + y + z \leq 78$, and their average $(w + x + y + z)/4 \leq 19.5$. Contradiction. ■

Problem 16: Assume for a contradiction that each of the n people at the party shook a different number of hands. Since the greatest number of hands a person would have shaken is $n - 1$ (everyone's but their own), all numbers of handshakes must lie in the range 0, 1, 2, $\ldots n - 1$. Since the size of that set is the same as the number of people at the party (n), it must be that for each number in that range, there must be someone at the party who shook that number of hands. But now we have a contradiction, because you cannot have one person shake $n-1$ hands and still have someone who shook 0 hands. $\blacksquare$

Chapter 1

Problem 2: $\{2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47\}$

Problem 6: To show that two sets are equal, we need to show that each is a subset of the other. That is, we show that every element in the set on the left is in the set on the right, and that every element in the set on the right is in the set on the left. We'll do those two things, in that order.
Let $x \in A \cup (A \cap B)$. Then $x \in A$ or $x \in A \cap B$. In either case, we have $x \in A$ and we are done.
Now let $x \in A$. The x will be in A union anything. In particular, $x \in A \cup (A \cap B)$. $\blacksquare$

Problem 12: Let $x \in A''$. Then $x \in U - A'$, which implies $x \in U$ and $x \notin A'$. But if $x \notin A'$, then $x \notin (U - A)$. Saying that $x \in (U - A)$ is saying $x \in U \wedge x \notin A$. So saying the negation of that is saying $x \notin U \vee x \in A$. Clearly $x \notin U$ is impossible, since U is the universal set. Thus it must be the case that $x \in A$.
Now let $x \in A$. We wish to show that $x \in A''$. But this proof is very similar to that above, so I'll leave it as an exercise for you to complete.particular, $x \in A \cup (A \cap B)$. $\blacksquare$

Problem 14: Let $x \in A \cap B$. Then $x \in A$, which implies that $x \in A$ union anything. In particular, $x \in A \cup B$. $\blacksquare$

Problem 20: **Theorem:** $A \subseteq B \leftrightarrow A \cap B' = \emptyset$.
Proof: ($\rightarrow$) Suppose that $A \subseteq B$. Then $x \in A \rightarrow x \in B$, which is the same as saying $x \in A \rightarrow x \notin B'$. But then $A \cap B'$ can have no elements, that is, $A \cap B' = \emptyset$.
($\leftarrow$) Now suppose that $A \cap B' = \emptyset$. This means that A and B' have no elements in common, so that any element of A cannot be in B'. Thus if x is some element of A, then $x \notin B'$ which implies $x \in B$. Thus A is a subset of B. $\blacksquare$

Problem 25: $A \times B = \{(1,2),(1,3),(2,2),(2,3)\}$

Problem 26: This is false, as can be shown by considering $B \times A$ in the previous problem. $B \times A = \{(2,1),(2,2),(3,1),(3,2)\}$.

Problem 30: {HHH, HHT, HTH, HTT, THH, THT, TTH, TTT}.

Problem 32: This was proved in both classes, thank goodness! ■

Problem 40: These sets have no element common to all of them, so that $\cap_{i=1}^{4} A_i = \emptyset$. $\cup_{i=1}^{4} A_i$ is the union of all elements in these sets, which is {2, 3, 4, 8, 9, 12, 16, 19, 27, 32}

Problem 46: We'll use distributivity first to rewrite $(A \cup B) \cap (A' \cup B)$ as $((A \cup B) \cap A') \cup ((A \cup B) \cap B)$. Then we'll use distributivity inside each of those to write it as $(((A \cap A') \cup (B \cap A')) \cup ((A \cap B) \cup (B \cap B)))$. Associativity allows us to get rid of a bunch of those parentheses:
$= (A \cap A') \cup (B \cap A') \cup (A \cap B) \cup (B \cap B)$
$= \emptyset \cup (B \cap A') \cup (A \cap B) \cup B.$
$= (B \cap A') \cup (A \cap B) \cup B.$
We use commutativity on the middle term:
$= (B \cap A') \cup (B \cap A) \cup B$
and the use distributivity to regroup the two leftmost terms:
$= B \cap (A' \cup A) \cup B$
$= (B \cap U) \cup B = B \cup B = B.$
And that's about as simple as they get!

Another way to do this is to use distributivity from the beginning, but to *group* the terms instead of breaking them open. We get $(A \cup B) \cap (A' \cup B) = (A \cap A') \cup B$. Then we observe that $(A \cap A') = \emptyset$, so that we can rewrite $(A \cap A') \cup B$ as $\emptyset \cup B$ which is clearly B.

Problem 48: This was proved in both classes, thank goodness!

Problem 58: This is easily shown not to hold in general, by letting $A = B = C = \{1\}$. Then $(A - B) - C = \emptyset$, but $A - (B - C) = \{1\}$.

Chapter 2

Problem 2: By plugging in values for x we find that at $x = 11$, $2^x = 2048$ and $2x^3+x^2+1 = 2784$, so 2^x is still smaller. But when $x = 12$, $2^x = 4096$ and $2x^3+x^2+1 = 3601$, so 2^x is greater. It will stay greater forever after that (you don't need to prove this), so $N = 12$.

Problem 4: Both of these are true when x and y are integers. To make the floor expression false, let $x = y = 0.5$, and to make the ceiling expression false, let $x = y = 0.1$.

Problem 8 It is easy to analyze this relation if we realize that the expression $|x - y| < 1$ is the same as saying x and y are distance less than 1 apart on the number line. To see that this is not transitive we can let $x = 1, y = 1.8$ and $z = 2.6$. Then $x \sim y$ and $y \sim z$ but $x \nsim z$. It is reflexive, because any element is distance 0 from itself. It is symmetric because $|x - y| = |y - x$, and since it is symmetric, it cannot be antisymmetric.

Problem 10 It is an equivalence relation, becuase it is reflexive, symmetric and transitive. When you realize that two triangles are similar if they have *the same* angles, then it is obviosu that this relation is reflexive, symmetric and transitive.

Problem 12 The equivalence class of any real number is all of $\mathbb{R}$.

Problem 14 We need to show that $\subseteq$ is reflexive, antisymmetric and transitive. It is reflexive, because any set is obviously a subset of itself. Transitivity is clear from the definition of subset. To see that it is antisymmetric, we need to show that for any two sets A and B with $A \neq B$, if $A \subseteq B$ then $B \nsubseteq A$. To show this, consider two sets A and B and suppose that $A \neq B$ and $A \subseteq B$. If it was the case that $B \subseteq A$,then by definition of set equality, we would have $A = B$. But that contradicts our assumption.

Problem 20 An order relation must be antisymmetric while an equivalence relation must be symmetric. Thus an order relation cannot be an equivalence relation.
Actually, there is one exception to this rule: If the set on which we have the relation has just one element, then it *can* be both symmetric and antisymmetric.

Problem 24 One example is the relation "is a first cousin of." Another example can be created by simply building a set of ordered pairs to satisfy the requirements of the problem. For example, I can let $A = \{1, 2\}$ and the relation be $\{(1, 2), (2, 1)\}$.

Problem 26 We verify the three conditions for a relation to be an equivalence relation. This is reflexive because any computer can communicate with itself. It is symmetric because if A can communicate with B then clearly B can communicate with A. It is transitive, because if A can communicate with B and B can communicate with C, then it is possible for A to communicate with C, by connecting through B.

Problem 32 It is obvious that this is an equivalence relation, because what it really says is that two elements are related if they have *the same* square. The equivalence classes are the sets of the form $\{0\}$ and $\{a, -a\}$ for $a \neq 0$.

Problem 34 There are $n(n-1)(n-2)\cdots(n-m+1)$ 1-1 functions from A to B, as discussed in class.

Problem 36 We wish to show that for every $y \in C$ there is some $x \in A$ such that $(g \circ f)(x) = y$. Since g is surjective, we know that there is some $b \in B$ such that $g(b) = y$, and since f is surjective, there is some $x \in A$ such that $f(x) = b$. This gives the desired a, for then $g(f(a)) = g(b) = y$.