Discrete Mathematics **Exam II Solutions**

1. Compute each of the expressions below, where $A = \{\varnothing, 1, 2, 3\}$ and $B = \{2, 3, 4, 5\}$.
   a. $A \cup B$ $= \{\varnothing, 1, 2, 3, 4, 5\}$
   b. $A \cap B$ $= \{2, 3\}$
   c. $A \setminus B$ $= \{\varnothing, 1\}$

2. Compute the power set of each of the following sets.
   a. $\varnothing$ Answer: $\{\varnothing\}$
   b. $\{1, 2\} \cup \{2, 3\}$ Answer: $\{ \varnothing, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\} \}$
   c. $\{a, b\}$ Answer: $\{ \varnothing, \{a\}, \{b\}, \{a, b\} \}$

3. What is the cardinality of the power set of a set with *n* elements?
   2^n Know this!

4. What is $|\{x \in \mathbb{N} \mid -5 \le x \le 7\}|$? Your answer should be an integer.
   The notation "|A|" means "cardinality of A," where A is some set. (Recall that "cardinality" means "size.") So you are finding the size of the set of natural numbers between –5 and 7, that is, the size of the set
   $\{0, 1, 2, 3, 4, 5, 6, 7\}$, which is 8.

5. Define the following terms, related to functions:
   a. 1-1 A function is 1-1 if no two different elements of the domain map to the same element of the codomain
   b. onto A function is onto if every element of the codomain gets hit by some element of the domain.
   c. bijection A function is a bijection if it is both 1-1 and onto

6. Evaluate each of the following quantities:
   a. 50 (mod 7) Answer: 1, because 50 divided by 7 leaves a remainder of 1.
   b. 779^{639} (mod 7) Answer: You can reduce the base 779, mod 7, to 2, because 779 / 7 leaves a remainder of 2. Then, since the modulus, 7, is prime, Fermat's Little Theorem lets you reduce the exponent, mod 6, to 3. So you can compute 2^3 (mod 7), which is 1.
   c. $40^{8132469132467814678}$ (mod 8)
   Since $40 \equiv 0 \pmod 8$, this reduces to $0^{8132469132467814678}$ (mod 8), which is obviously 0.

7. Please state Fermat's Little Theorem
   If *p* is prime, and *a* is relatively prime to *p*, then $a^{p-1} \equiv 1 \pmod p$

8. Evaluate $\sum_{i=3}^{7}(i^2 + 2)$, and express your answer as an integer
   This sum asks you to plug in the values 3, 4, 5, 6, 7 and add them together. You don't need a formula for that. You get $(3^2 + 2) + (4^2 + 2) + (5^2 + 2) + (6^2 + 2) + (7^2 + 2) = 145$.

9. Evaluate $\sum_{i=1}^{100} 4 \cdot 3^i$. Your answer should be in closed form. That is, it shouldn't have any "..." in it.

For this one, you do want to use a formula. This is the sum of a geometric series, with constant ratio 3. (The terms are $4\times3 + 4\times3^2 + ... + 4\times3^{100}$.) The formula for such a sum is (next – first) / (ratio – 1). The next term would be 4×3^{101}, the first term is 12, and the ratio is 3. Plugging in to the formula, you obtain:

$(4\times3^{101} - 12) / 2$, which is the answer.

10. Use the Euclidean algorithm to find the greatest common divisor of 43 and 32.

43	=	1×32	+	11
32	=	2×11	+	10
11	=	1×10	+	1
10	=	10×1	+	0

So the gcd is the last non-zero remainder, 1.

11. Use your work in the previous problem to write 1 as an integer combination of 43 and 32.

Start with the row containing the "1" as the remainder, and solve for "1."

1 = 11 – 1×10

On each step thereafter, substitute for the remainder in the previous row, then distribute to obtain that remainder as an integer combination of the dividend and divisor in that row:

Substitute for 10:	1	=	11	–	1×(32 – 2×11)
Distribute:		=	3×11	–	1×32
Substitute for 11:		=	3×(43 – 1×32)	–	1×32
Distribute:		=	3×43	–	4×32

And that's the answer.

12. Prove that if $d \mid a$ and $d \mid b$, then $d \mid (a + b)$

$d \mid a \rightarrow a = kd$ for some integer k

$d \mid b \rightarrow b = ld$ for some integer l

Then $a + b = kd + ld = d(k + l)$, which implies $d \mid a + b$.

13. a. Prove by induction that for all values of $n \geq 1$: $6 + 12 + ... + 6n = 3n(n+1)$

Base Case: $n = 1$. The left side is just 6, and the right side is $3\times1\times(1 + 1)$, which is also 6. ✓

Induction Hypothesis: Suppose it is true for some value k.

That is, suppose that $6 + 12 + 18 + ... + 6k = 3k(k + 1)$

Induction Step: Show that it is true for $k + 1$.

That is, show that $6 + 12 + 18 + ... + 6k + 6(k + 1) = 3(k + 1)(k + 2)$

To show this, consider the left hand side:

$6 + 12 + 18 + ... + 6k + 6(k + 1)$

= (by the I.H.) $3k(k + 1) + 6(k + 1) = (k + 1)(3k + 6) = 3(k + 1)(k + 2)$, which is what we wanted to show. This completes the induction. ✓

13. b. Prove that a positive integer is a multiple of 3 if and only if the sum of its digits is a multiple of 3.

An integer is a string of digits $d_k ... d_3d_2d_1d_0$. This integer equals $d_k\times10^k + \cdots + d_2\times10^2 + d_1\times10^1 + d_0$. Since $10 \equiv 1 \pmod 3$, each power of 10 will be congruent to 1 (mod 3). So, mod 3, this integer is congruent to $d_k\times1^k + \cdots + d_2\times1^2 + d_1\times1^1 + d_0$, which is just the sum of the digits. Thus an integer is congruent to the sum of its digits (mod 3), which implies that if one is divisible by 3 (congruent to 0) then the other is also.

14. Prove that if p and q are two prime numbers with $\gcd(p + q, p - q) > 2$, then $p = q$.
Let's prove the contrapositive of this statement: Suppose that p is not equal to q. (We then wish to show $\gcd(p + q, p - q) \leq 2$.)

If p is not equal to q, then the only common divisor they could have is 1, since they are both prime. Thus $\gcd(p, q) = 1$, which implies that $\gcd(2p, 2q) = 2$.

Now let $d = \gcd(p + q, p - q)$. Then $d \mid (p + q)$ and $d \mid (p - q)$, which implies d divides the sum and difference of those quantities. That is, $d \mid 2p$ and $d \mid 2q$. But since $\gcd(2p, 2q) = 2$, d must be ≤ 2, which is what we wanted to prove.

Discrete Mathematics **Name**__________________________________

Exam II - Makeup **November 30, 2005**

This test has 13 questions. Please make sure that you have them all.

1. Compute each of the expressions below, where $A = \{1, 2, 3, 4, 5, 6\}$ and $B = \{2, 3, 4, 5\}$.
   a. $A \oplus B$

   b. $A \cap B$

   c. $A \setminus B$

2. Compute the power set of each of the following sets.
   a. $\varnothing$

   b. The power set of $\{1\}$. (That is, you should find the power set of the power set of $\{1\}$.)

3. Construct a set A with the property that $|\mathcal{P}(A)| = 4 \times |A|$, and justify your answer.

4. What is $|\{x \in \mathbb{N} \mid x^2 = x\}|$? Your answer should be an integer.

5. Define the following terms, related to functions:
   a. 1-1

   b. onto

   c. bijection

6. Evaluate each of the following quantities:

a. 50 (mod 8)

b. 553^{443} (mod 5)

c. $41^{8132469132467814678}$ (mod 8)

7. Please state Fermat's Little Theorem, from memory.

8. Evaluate $\sum_{i=6}^{8} (i^3 - 2)$, and express your answer as an integer

9. Evaluate $\sum_{i=10}^{100} 3^i$. Your answer should be in closed form. That is, it shouldn't have any "..." in it.

10. Use the Euclidean algorithm to find the greatest common divisor of 52 and 41.

11. Use your work in the previous problem to write 1 as an integer combination of 52 and 41.

12. Prove that if $d \mid a$ and $d \mid (a + b)$, then $d \mid b$

13. **Do one of the following two problems:**

a. Prove by induction that for all values of $n \geq 1$: $4 + 8 + 12 + \ldots + 4n = 2n(n+1)$

b. Let the sequence $\{f_n\}$ (these are the Fibonacci numbers) be defined for all $n \geq 0$ by $f_0 = 0$, $f_1 = 1$, and for $n \geq 2$, $f_n = f_{n-1} + f_{n-2}$. What is the value of f_{10000} (mod 7)?