Exam 1 Solutions

1. Fill in the rest of this table.

Name of Syllogism	**Symbolic Representation**
Addition	$p \rightarrow (p \vee q)$
Modus ponens	$[p \wedge (p \rightarrow q)] \rightarrow q$
Modus tollens	$[\neg q \wedge (p \rightarrow q)] \rightarrow \neg p$
Hypothetical Syllogism	$[(p \rightarrow q) \wedge (q \rightarrow r)] \rightarrow (p \rightarrow r)$
Disjunctive syllogism	$[\neg p \wedge (p \vee q)] \rightarrow q$
Resolution	$[(p \vee q) \wedge (\neg p \vee r)] \rightarrow (q \vee r)$

2. Assuming that A → B, ¬C → D and C → A are all true, show that ¬D → B is true.
The contrapositive of ¬C → D is ¬D → C. *We then have the following three implications:* ¬D → C, C → A *and* A → B. *Then two applications of hypothetical syllogism gives us* ¬D → B.

3. Given that A ∨ B, ¬A, ¬X, P ∨ A and ¬P ∨ B are all true, show that ¬(B → X) is true.
¬A and (A ∨ B) *give us, by disjunctive syllogism,* B. ¬X *is given, so that we have* (B ∧ ¬X), *which is logically equivalent to* ¬(B → X), *being the rule for negating implication.*

4. Use truth tables to show that $p \rightarrow q$ and $\neg p \vee q$ are logically equivalent.

p	q	$p \rightarrow q$	$\neg p$	$\neg p \vee q$
T	T	T	F	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

In the truth table to the left, the columns for the expressions $p \rightarrow q$ *and* $\neg p \vee q$ *are identical, indicating that they are logically equivalent.*

5. Negate each of the following expressions:
   a. $p \rightarrow q$ *becomes* $p \wedge \neg q$
   b. $(p \oplus q) \vee (p \leftrightarrow q)$ *becomes* $(p \leftrightarrow q) \wedge (p \oplus q)$
   c. $\forall t\, P(t)$ *becomes* $\exists t\, \neg P(t)$

6. Let $A = \{1, 2, \{3, 4\}, 5\}$ and $B = \{2, 3, 4\}$. True or False (if you say "False", say why):
   a. $A \subseteq B$ *False, because 1 is an element of A, but not of B*
   b. $B \subseteq A$ *False, because 3 is an element of B, but not of A*
   c. $\{3, 4\} \subseteq A$ *False, because 3 is not an element of A*
   d. $\{3, 4\} \subseteq B$ *True.*
   e. $\{3, 4\} \in A$ *True.*
   f. $\{3, 4\} \in B$ *False, because there is no such element as {3, 4} in B.*

7. Compute each of the expressions below, where $A = \{1, 2, \{3, 4\}, 5\}$ and $B = \{2, 3, 4\}$.
   a. $A \cup B$ = *{1, 2, 3, 4, 5, {3, 4}}*
   b. $A \cap B$ = *{2}*
   c. $A - B$ = *{1, {3, 4}, 5}*
   d. $A \oplus B$ = *{1, 3, 4, 5, {3, 4}}*

8. Compute the power set of each of the following sets.
   a. ∅ *{∅}*
   b. {1} *{∅, {1}}*
   c. {2, 3, 4} *{∅, {2}, {3}, {4}, {2, 3}, {2, 4}, {3, 4}, {2, 3, 4}}*
   d. {∅} *{∅, {∅}}.*

9. Use membership tables to prove that for all sets A and B, $(A\cup B) - (A\cap B) = (A - B) \cup (B - A)$

A	B	A∪B	A∩B	(A∪B) −(A∩B)	A −B	B − A	(A −B) ∪ (B − A)
1	1	1	1	0	0	0	0
1	0	1	0	1	1	0	1
0	1	1	0	1	0	1	1
0	0	0	0	0	0	0	0

The 5th and 8th columns are identical, indicating that the expressions give the same set.

10. Use membership tables to prove that for all sets A, $(A \oplus \mathbf{U}) = \overline{A} - \varnothing$. Here "**U**" stands for the universal set.

A	∅	U	(A ⊕ U)	$\overline{A}$	$\overline{A}$ − ∅
1	0	1	0	0	0
0	0	1	1	1	1

The 4th and 6th columns are identical, indicating that the expressions give the same set.

11. Suppose we have some variant of the standing bosses game with the following properties:
   a. There is some unique start position
   b. A position consists of a row of players, fixed in number, each either sitting or standing
   c. There is a rule which, at each step, determines the next position from the current position alone, without using any randomness or any outside input.

   Prove that such a game on n players must either continue indefinitely, or stop within 2^n steps.

If the game stops within 2^n steps, then we are done. If the game continues beyond 2^n steps, then there must be some repeated position, because 2^n is the number of possible positions among n players. If a position is repeated, then property c. implies this position will repeat itself indefinitely, meaning that the game will never terminate.

12. Prove by mathematical induction that $\forall n \geq 1$, $4 + 8 + 12 + \cdots + 4n = 2n(n + 1)$

Base case: n = 1

The sum on the left hand side is 4, and the expression on the right gives $2\times1\times(1+1)$*, which is also 4.*

Induction Hypothesis: Suppose the theorem is true for some fixed value k. That is, assume
$4 + 8 + 12 + \cdots + 4k = 2k(k + 1)$.

Induction Step: We wish to show it is true for k + 1. That is, we wish to show:
$4 + 8 + 12 + \cdots + 4k + 4(k + 1) = 2(k + 1)(k + 2)$.

Consider the left hand side of this equation:
$4 + 8 + 12 + \cdots + 4k + 4(k + 1)$
$= [4 + 8 + 12 + \cdots + 4k] + 4(k + 1)$
$=$ *(by the induction hypothesis)* $[2k(k + 1)] + 4(k + 1)$
$= (k + 1)(2k + 4)$
$= 2(k + 1)(k + 2)$
Which is what we wanted to show.

13. Extra Credit --- The "Who's the Boss" sequence: 1, 2, 1, 3, 1, 2, ...

a. Give the first 30 terms

1 2 1 3 1 2 1 4 1 2 1 3 1 2 1 5 1 2 1 3 1 2 1 4 1 2 1 3 1 2

b. What is the sum of the first 1024 terms?

Sum of the first 1 term = 1
Sum of the first 2 terms = 3
Sum of the first 4 terms = 7
Sum of the first 8 terms = 15
Sum of the first 16 terms = 31
If this pattern continues, and you can show that it does by considering the way the sequence is generated, then the sum of the first 1024 terms would be $2\times1024 - 1 = 2047$