

Discrete Mathematics --- Day 30 --- November 5, 2004

11-05-04 Quiz Solution:

A. How many ways can you arrange 11 quarters with 3 pockets (pockets may be empty)?

Start with "P" and find how many anagrams there are of 11 quarters QQQQQQQQQQQ and 2 pockets PP.

$$\frac{13!}{2! \times 11!} = \frac{13 \times 12}{2} = 78$$

B. How many ways can you arrange 11 quarters with 3 pockets where each pocket has at least 1 quarter?

Reserve 3 quarters one for each pocket, next start with "P" and find how many anagrams there are of 8 quarters QQQQQQQQ and 2 pockets PP.

$$\frac{10!}{2! \times 8!} = \frac{10 \times 9}{2} = 45$$

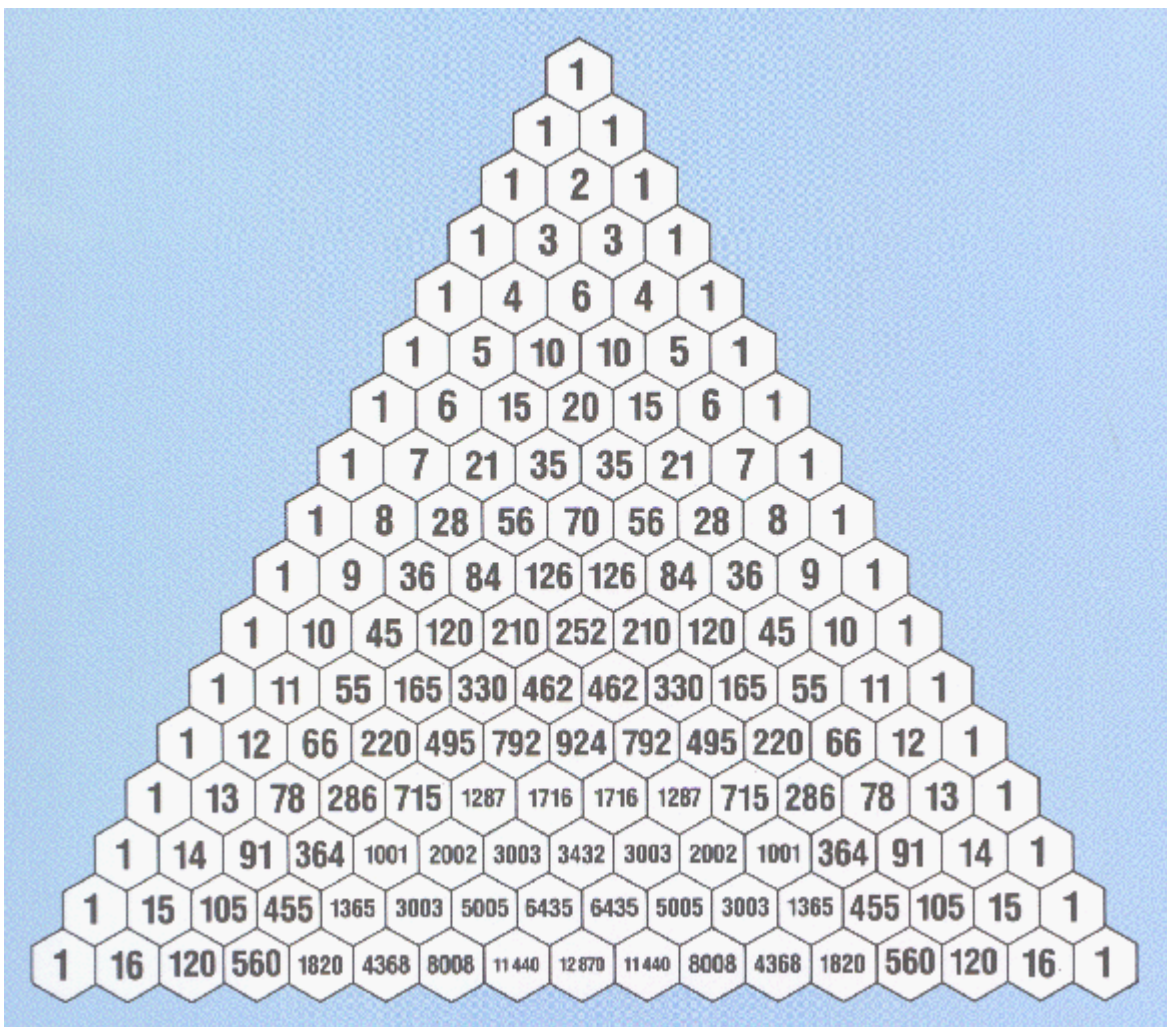
C. How many ways can you arrange 11 quarters with 3 pockets where the 1st pocket must have at least 1 quarter, and the 2nd must have at least 3 quarters?

First, reserve 4 quarters, 1 for the 1st pocket and 3 for the 2nd pocket. Next, start with "P" and find the number of anagrams of 7 quarters QQQQQQQ and 2 pockets PP.

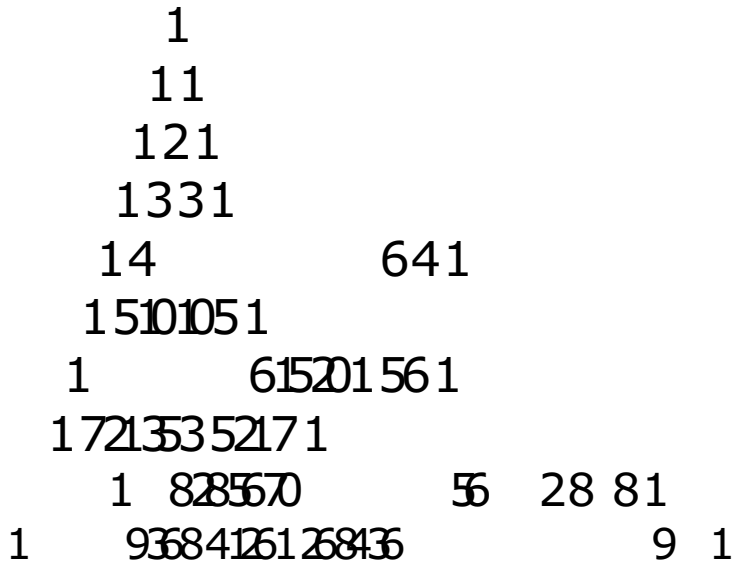
$$\frac{9!}{2! \times 7!} = \frac{9 \times 8}{2} = 36$$

| | | | |
|---|---|----|----|
| 1 | 1 | 1 | 1 |
| 1 | 2 | 3 | 4 |
| 1 | 3 | 6 | 10 |
| 1 | 4 | 10 | 20 |

This grid turned 45° clockwise



Pascal's Triangle:



This triangle has all the "Choose Numbers" in it:

$${}^7\text{Choose } 2 = 21$$

$$\binom{7}{2} = C(7, 2) = {}_7C_2$$

$$\binom{6}{3} = 20$$

$$\binom{5}{2} = 10 = \binom{5}{3}$$

If it takes 8 steps to get from point A to point B on a grid that includes 4 rights, *R*, and 4 downs, *D*. There will be 70 ways to walk from point A to B

because the number of anagrams of *R R R R D D D D* is $\frac{8!}{4!(4!)} = \binom{8}{4} = 70$.

Going to the 8th row and 4th position (starting with 1 as the 0 position) on the Pascal's Triangle the answer 70 is located.

To find $\binom{n}{k}$ in Pascal's Triangle, go to the n^{th} row (each row starts with 1 then is immediate followed by n) and find the k^{th} entry, keeping in mind that the "1" is the 0^{th} entry.

Prove that
$$\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}$$

Proof:

$$\begin{aligned} & \frac{n!}{k!(n-k)!} + \frac{n!}{(k+1)!(n-k-1)!} \\ &= \frac{n!(k+1)}{(k+1)!(n-k)!} + \frac{n!(n-k)}{(k+1)!(n-k)!} \\ &= \frac{n!*k + n! + n!*n - n!*k}{(k+1)!(n-k)!} \\ &= \frac{n!(1+n)}{(k+1)!(n-k)!} \\ &= \frac{(n+1)!}{(k+1)!(n-k)!} \\ &= \binom{n+1}{k+1} \end{aligned}$$

Sum of numbers in the n^{th} row is 2^n (Pascal's Triangle)

Binomial Theorem

For a natural number n , $(a + b)^n = \sum_{k=0}^n \binom{n}{k} a^{n-k} b^k$

$$\begin{aligned} \text{Ex.1 } (a + b)^3 &= (a + b)(a + b)(a + b) \\ &= (a + b)(a^2 + 2ab + b^2) \\ &= a^3 + 2a^2b + b^3 + a^2b + 2ab^2 + b^3 \\ &= a^3 + 3a^2b + 3ab^2 + b^3 \end{aligned}$$

$$\text{Ex.2 } (x + y)^6 = 1x^6y^0 + 6x^5y^1 + 15x^4y^2 + 20x^3y^3 + 15x^2y^4 + 6x^1y^5 + 1x^0y^6$$

“Sigma notation” plugs in the various values of k from 0 to n , and add up the terms you obtain.

Expand $(a + b + c)^4$ by using 3 natural numbers whose sum is 4.

| <u>#a's</u> | <u>#b's</u> | <u>#c's</u> |
|-------------|-------------|-------------|
| 4 | 0 | 0 |
| 0 | 4 | 0 |
| 0 | 0 | 4 |
| 3 | 1 | 0 |
| 3 | 0 | 1 |
| 1 | 3 | 0 |
| 0 | 3 | 1 |
| 0 | 1 | 3 |
| 1 | 0 | 3 |
| 2 | 2 | 0 |
| 2 | 0 | 2 |
| 0 | 2 | 2 |
| 2 | 1 | 1 |
| 1 | 2 | 1 |
| 1 | 1 | 2 |

The expansion that results is:

$$\begin{aligned} a^4 + 4a^3b + 4a^3c + 6a^2b^2 + 12a^2bc + 6a^2c^2 + 4ab^3 + 12ab^2c + 12abc^2 + 4ac^3 + b^4 + 4b^3c \\ + 6b^2c^2 + 4bc^3 + c^4 \end{aligned}$$

Where anagrams are used to obtain the coefficients. For example, the coefficient of ab^2c is 12, because $12 = 4! / (1! \times 2! \times 1!)$