Discrete Mathematics----Day 15----September 29, 2004

<u>Theorem (*Ultimate Theorem*)</u>: $\forall n \in \mathbf{Z}$, $n > 1$, the $n$-disk towers of Hanoi cannot be solved in fewer than $2^n$-1 moves.

Logic: Theorem *2* → Theorem *3* → Theorem *4* …

Theorem 2: The *2* disk…$2^2$-1 = 3 moves
Theorem 3: The *3* disk…$2^3$-1 = 7 moves
Theorem 4: The *4* disk…$2^4$-1 = 15 moves
Theorem 5: The *5* disk…$2^5$-1 = 31 moves

This is really infinite many theorems. BUT DO NOT PANIC!!! There is an easy, slick technique for proving them all at once.

<u>Dominoes</u>

- Infinitely many dominoes
- Require each domino to be so situated that if it falls, then the next one will fall also.
- Knock over the 1st one in such a way that it knocks over the second one

*Principle* - If we meet these two requirements then all dominoes will fall

*Principle of the Least Natural Number*: every non-empty set of numbers has a least element

Set of Natural Numbers whose digits all add to equal *9*

{9, 18, 27, 36, 45 …}

<u>Proof of Theorem (*Ultimate Theorem*)</u>:

- Part 1: Show dominoes are close enough together
- Part 2: Knock over 1st domino

P($n$) = the $n$-disk cannot be solved in fewer than $2^n$-1 moves

Dominoes = P(*1*), P(*2*), P(*3*), P(*4*) …

Ultimate Theorem (*Restated*): $\forall n \geq 1, n \in \mathbf{Z}$, P($n$)
That is, P($1$)$\wedge$P($2$)$\wedge$P($3$)$\wedge$P($4$)$\wedge$ …

Domino Falls: Showed P($k$) to be true
Dominoes are close enough: P($k$) $\rightarrow$ P($k$+1) ($k \geq 1$)

Part 1:

- Assume P($k$) this means $k$-disk tower cannot be solved in less than < $2^k$-1 moves.

- To solve a $k$+1 disk tower, we must 1st move the top $k$ disks from one peg to another. Before we can move the bottom ($k$+1) disks.

- By the hypothesis P($k$), this takes already $2^k$-1 moves. Then we move the bottom disk one more move. Then we move the top $k$ disks back atop the bottom disk. Again by P($k$), this takes $2^k$-1 more moves. So the number moves must be at least $2^k-1+1+2^k-1=2^{k+1}-1$, which is P($k$+1)

Part 2: Show 1st domino falls this is true because a 1 disk tower takes $2^1-1=1$ move, which is P(1)

This completes our proof by induction, that all infinitely many of our theorems are true.