Exercises — Binomial Coefficients

Warm-up Problems:

1. Jack has 7 kinds of flowers in his garden and wishes to give Jill a bouquet with 3 of them. In how many ways can this be done?

2. Expand $(k + l)^3$ using Pascal's triangle

3. How many different 5-card hands can be dealt from a standard 52-card deck?

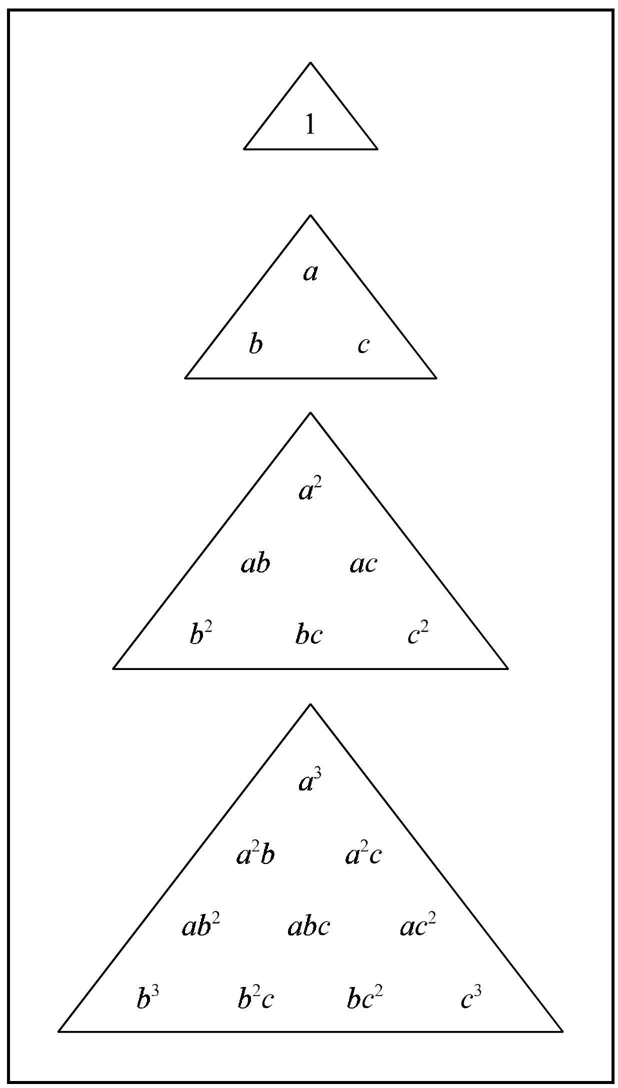
4. What is the probability that you win the **Cash5** game, which involves selecting 5 numbers out of 34, and you have to match all 5 of them to win?

5. What is the remainder when $\binom{1400}{69}$ is divided by 11?

6. Count the Ways.

Presentation Problems:

7. In this problem we will explore a Pascal's Triangle-like method for solving part f. of Handout #3.
   a. Expand $(a + b + c)$ to the first few powers, say from the 0th power to the 3rd power
   b. Now arrange the *coefficients* of the terms of your expansions into triangular arrays as shown to the right.
   c. If your triangular arrays of coefficients were arrays of tennis balls, then they would neatly stack one on top of the other, 1 - 3 - 6 - 10 - ..., forming a nice tetrahedral structure. Try it! We have plenty of tennis balls.
   d. Now, if you write your coefficients on the tennis balls, you would be completely justified in calling your structure Pascal's Tetrahedron. Do you see why?
   e. Generalize some of the patterns we observed in Pascal's Triangle to Pascal's Tetrahedron. How many of your assertions can you prove?
   f. Use Pascal's Tetrahedron to expand $(a + b + c)^4$
   g. If we call the tetrahedron having n balls along an edge the nth tetrahedron, then how many balls are in the bottom triangle of the nth tetrahedron?
   h. How many balls are in the nth tetrahedron?



8. Some Lottery Questions
   a. In the **Cash5** game, you select 5 numbers out of 35, and win \$5.00 if you get exactly 3 of them correct. What is the probability of this happening?
   b. In the **Cash3** Lottery (not yet implemented) a player pays \$1 for a ticket, selects 3 numbers out of 25, and if he gets all 3 of them correct, he wins \$2,500. What do you think?

9. Bob goes to the ice cream shop to buy Alice and himself sundaes. When he gets there, he realizes that he forgot what sort of sundae Alice wanted! All he remembers is that she wanted 3 different flavors of ice cream (the shop offers 14) and 2 different kinds of topping (the shop offers 5). Now, he knows it is important to Alice that he gets the right sundae, so he decides to get one of each possible type. How many sundaes does poor Bob have to buy?

10. How many 4-digit numbers are there whose digits appear in strictly increasing order?

11. How many different sequences of "H" and "T" are there which have exactly 3 H's and 7 T's? (Think of these as coin tosses.) "HTTHTTHTTT" is one such way.

12. Consider the remainders of the terms of the 120^{th} row of Pascal's Triangle when divided by 7
    a. How many of them are non-zero?
    b. Which terms are non-zero?
    c. How many terms of the 5000^{th} row are non-zero?

13. Some Coefficient Questions:
    a. What is the coefficient of a^4b^3 in the expansion of $(a + b)^7$?
    b. What is the coefficient of a^4b^3 in the expansion of $(2a + 3b)^7$?
    c. What is the coefficient of $a^3b^3c^3$ in the expansion of $(a + b + c)^9$?

Extension Problems:

14. The new governor of a certain state wishes to implement a new lottery called the "**Kash3**" game. He wants the game to require the selection of 3 numbers out of some large pool of numbers, so that the probability of getting all 3 of them correct is less than 1 in a million. But he doesn't want the pool to be too large, because then fewer people will play. What is the smallest size the pool can be so that the chances of getting all 3 correct is less than, or equal to, 1 in a million? (*The answer is 183*)

15. If you were asked to find the remainder when $\binom{200}{36}$ is divided by 6, you might initially be concerned that our trick won't apply, since 6 is not prime. But then you would realize that $6=2\times3$, so you could use the trick for 2 and 3, and then use, well, another trick to figure out what the remainder would be if we had divided by 6. Show that this remainder is 4.

16. Another way to think of the binomial theorem is as follows. When expanding $(a + b)^n$, start with the term $\dfrac{1 \cdot a^n \cdot b^0}{0!}$ and iterate the following procedure: To obtain each term from the previous, multiply the coefficient by the current exponent of a, decrease the exponent on a by 1 and increase the exponent on b by 1, and increase the number being "factorialed" in the denominator by 1. Thus if $n = 4$, the first three terms would be $\dfrac{1 \cdot a^4 \cdot b^0}{0!} + \dfrac{4 \cdot a^3 \cdot b^1}{1!} + \dfrac{12 \cdot a^2 \cdot b^2}{2!}$.

a. Complete the expansion of $(a + b)^4$. Isn't it interesting the way this procedure stops itself?

b. Try it again for the expansion of $(1 + x)^5$. Note that either the "1" or the "x" could play the role of "a" as described above. It wouldn't hurt to try it both ways, but at least be sure to try it in the case where "1" plays the role of "a," so that the first term is $\dfrac{1 \cdot 1^5 \cdot x^0}{0!}$ Note that in this case it is good to explicitly keep the exponent on the "1," even though 1 to any power is still 1.

c. Okay, that was all just warm-up. The real power of this version of the binomial theorem comes into play when the exponent is not a positive integer. For example, consider the expression $(1 - x)^{-1}$. Expanding this as above, we get the expression:

$$\frac{1}{1-x} = \frac{1 \cdot 1^{-1} \cdot (-x)^0}{0!} + \frac{-1 \cdot 1^{-2} \cdot (-x)^1}{1!} + \frac{2 \cdot 1^{-3} \cdot (-x)^2}{2!} + \frac{-6 \cdot 1^{-4} \cdot (-x)^3}{3!} + \cdots$$

which, as expected, simplifies to $(1 - x)^{-1} = 1 + x + x^2 + x^3 + \cdots$, the sum of a geometric series. This expression come in handy if you ever need to approximate an expression such as 1/0.98 by hand. Rewrite it as $1/(1 - .02)$ and use the first couple terms to get the approximation 1.0204. For this problem, your job is to do roughly the same thing to find the expansion of $1/(1 - x)^2$, and use it to approximate $1/0.97^2$.

d. It just keeps getting better. The same thing works fractional exponents! Use this method of binomial expansion to find an approximation of the square root of 1.05.

> How do I love thee? Let me count the ways.
> I love thee to the depth and breadth and height
> My soul can reach, when feeling out of sight
> For the ends of being and ideal grace.
> I love thee to the level of everyday's
> Most quiet need, by sun and candle-light.
> I love thee freely, as men strive for right.
> I love thee purely, as they turn from praise.
> I love thee with the passion put to use
> In my old griefs, and with my childhood's faith.
> I love thee with a love I seemed to lose
> With my lost saints. I love thee with the breath,
> Smiles, tears, of all my life; and, if God choose,
> I shall but love thee better after death.
> — *Elizabeth Barrett Browning*