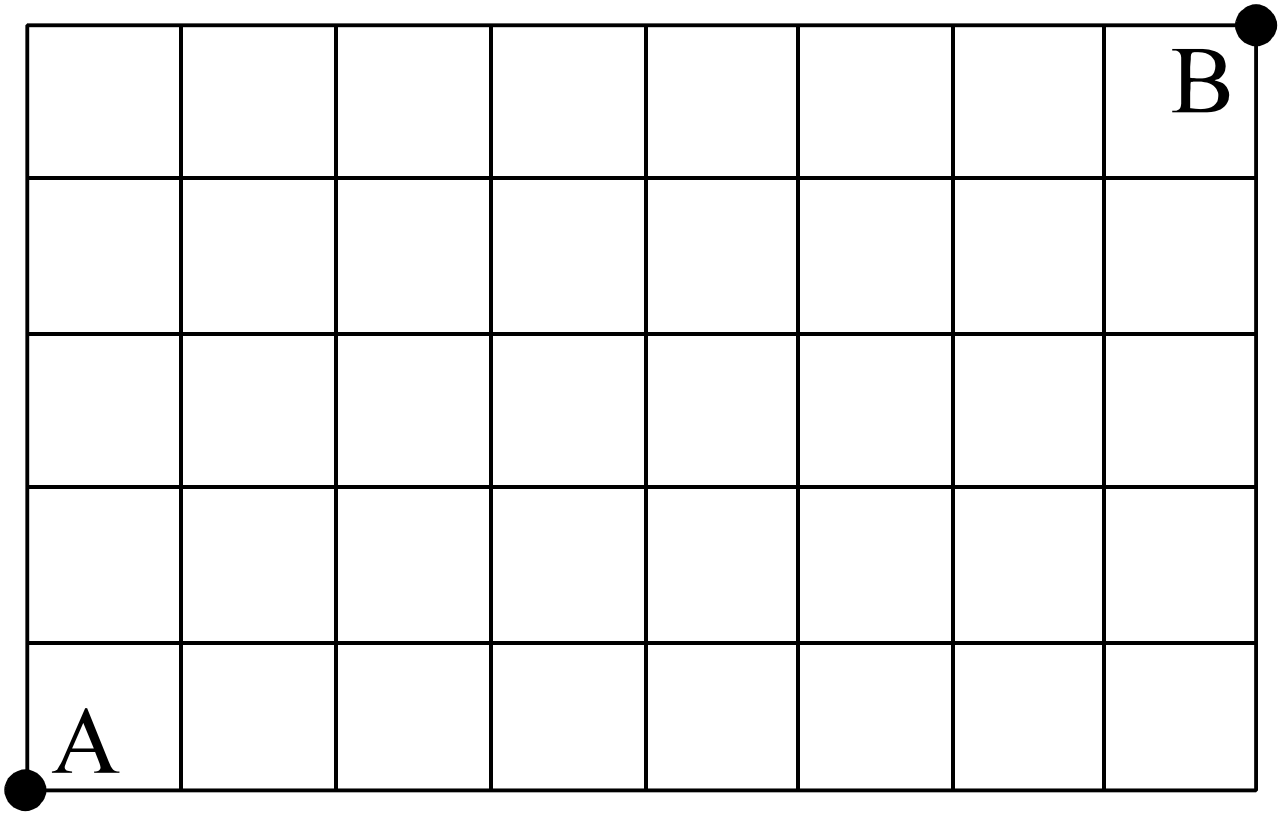
Some Questions

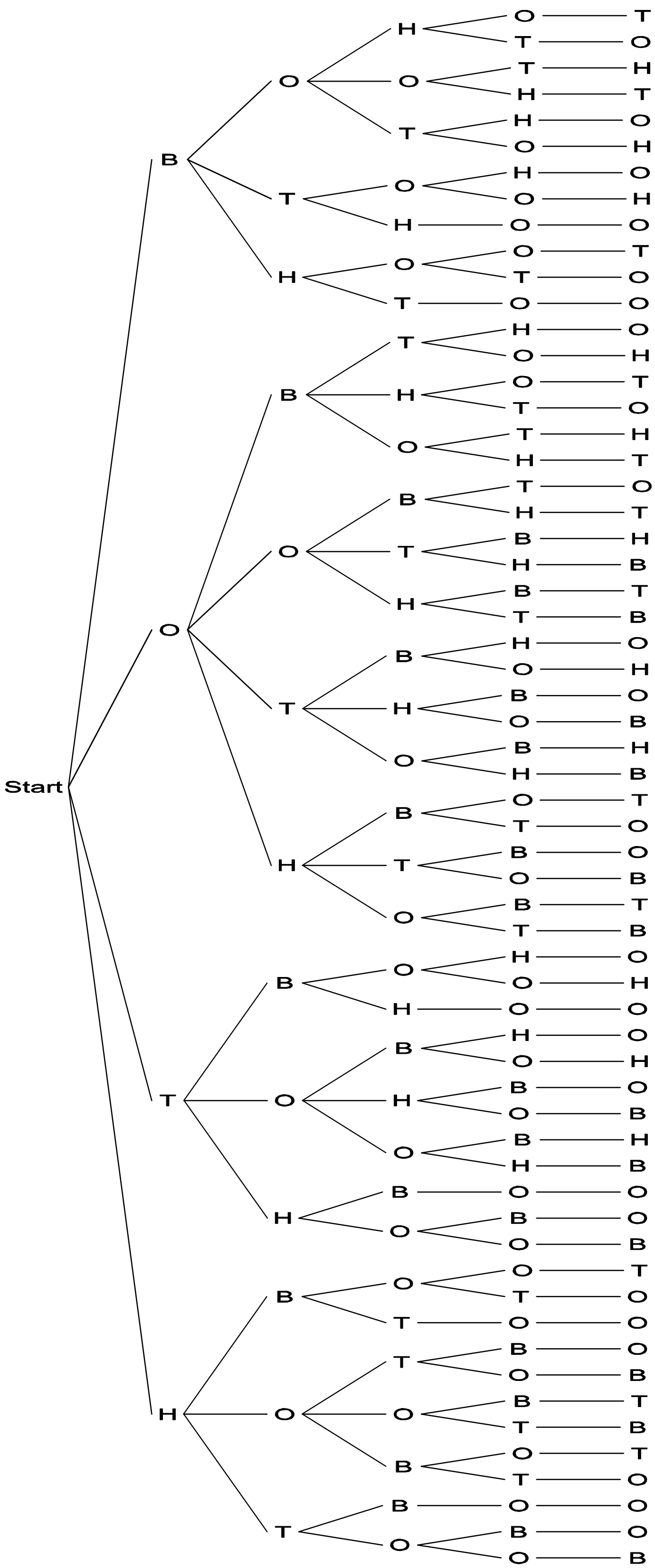
How many anagrams are there of “BOOTH”?

How many ways are there to walk from A to B on the grid below, without backtracking?

How many positive integral solutions are there to the equation $x + y + z = 30$?



BOOTH



Surely There is an Easier Way!

Counting Anagrams

For us, an "anagram" of a word is any rearrangement of the letters, whether it makes sense in English or not.

For example, the following are anagrams of STAR:

ARTS
RATS
TARS

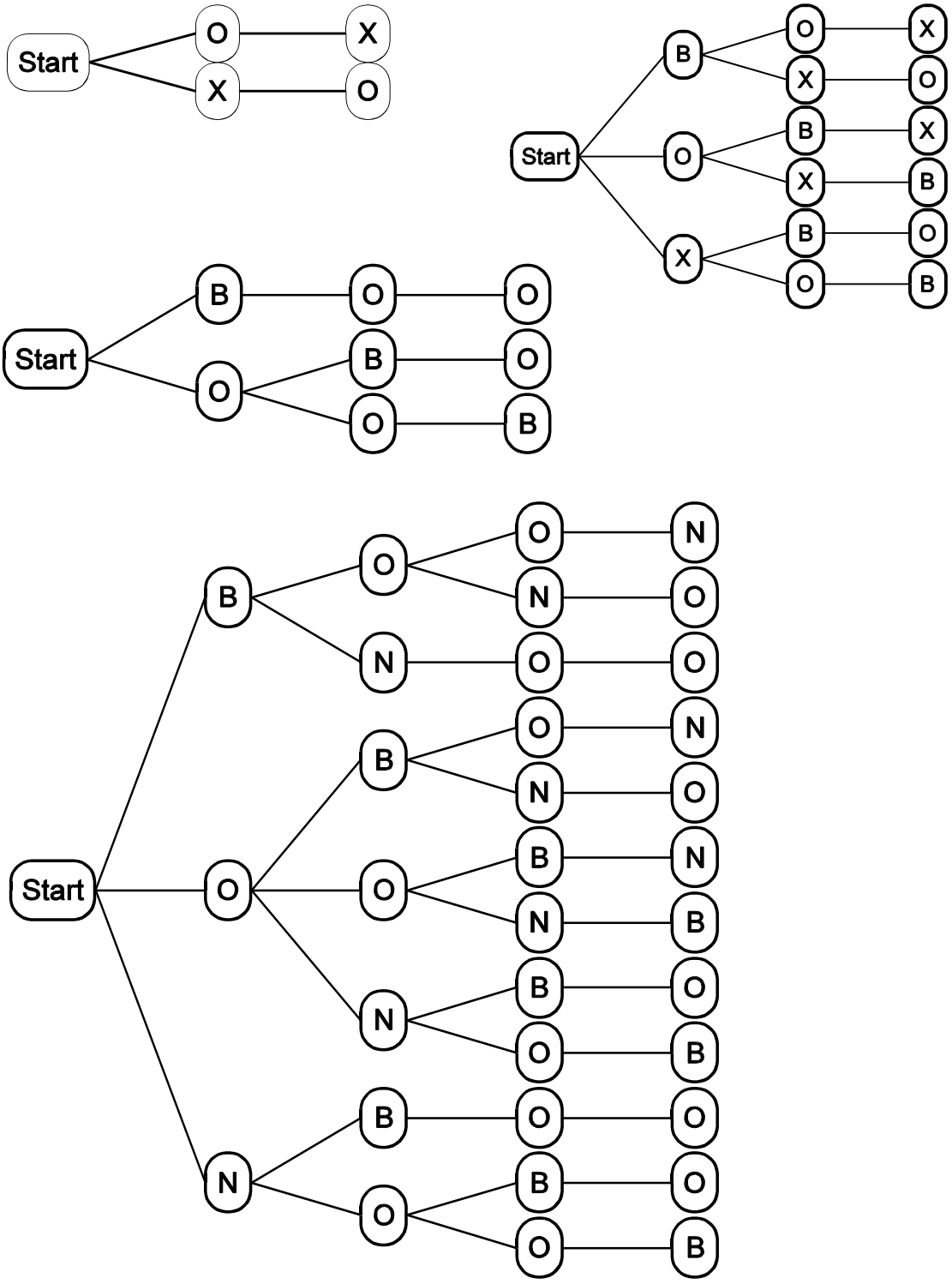
But so are:

STRA
SART
SATR
SRAT
SRTA
etc
⋮

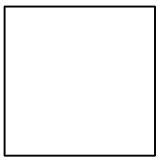
We are interested in the question of how many anagrams there are of any given word.

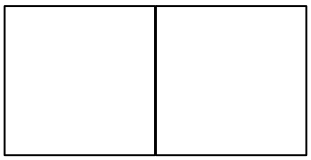
Let's Try Some Small Anagrams

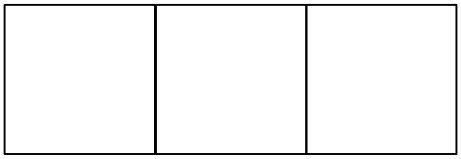
We have already done some of these

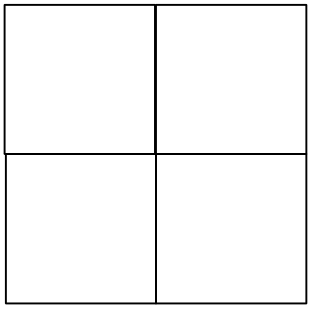


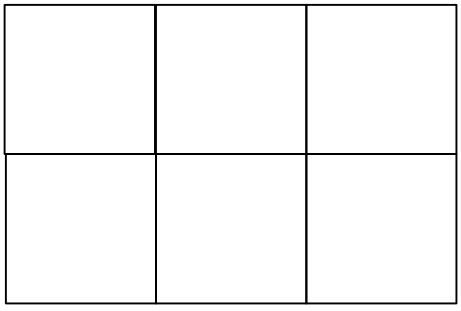
Let's Try Some Small Grids











Post-It Notes Activity Summary

An *anagram* is a rearrangement of the letters of a given word.

- The number of anagrams of DUEL is simply 4!, because the letters are all distinct

- The number of anagrams of DULL would be 4! if the two L's were different colors. But since they're the same color, we divide 4! by 2! to account for the L's interchanging places

- The number of anagrams of LULL would be 4! if the L's were all different colors. But since they're the same color, we divide 4! by 3! to account for the 3! ways the L's can be arranged

- The number of anagrams of LULU is 4! divided by (2!×2!) — One 2! for the L's and one 2! for the U's

Counting Anagrams

How many anagrams are there of each of the following words?

BETTER

REPAPER

SASSY

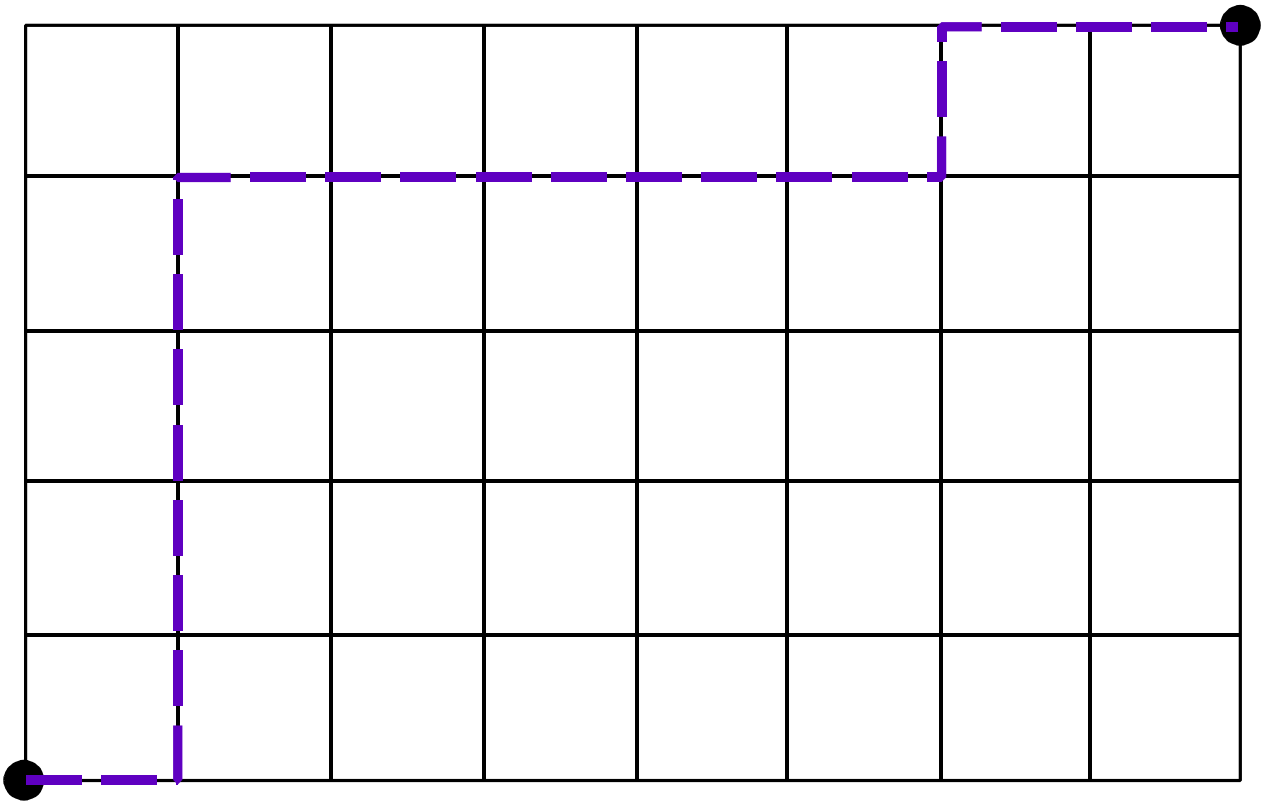
POWWOW

ITERATE

ANAGRAMS

Connection with Walking on a Grid

Here's one way to walk on the grid.

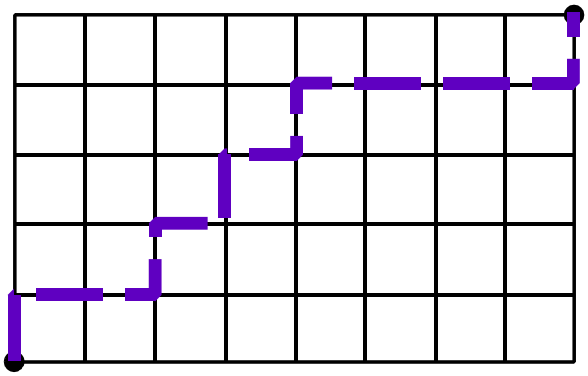


We can encode this route using "N" for north and "E" for east. The route becomes

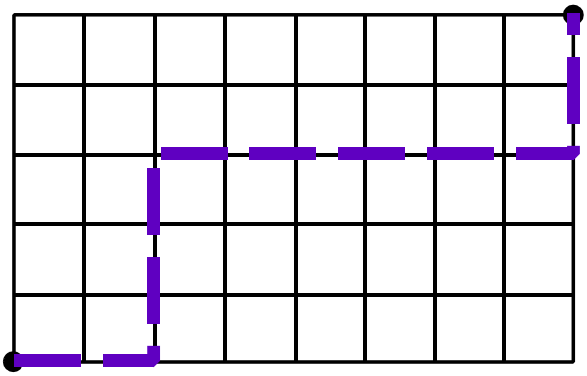
ENNNNEEEEENEE

What can you say about the string of letters +-+++above?

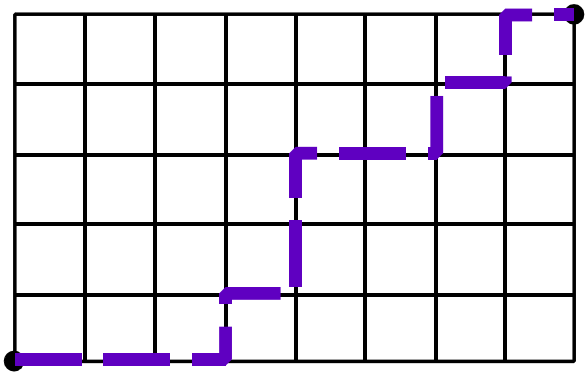
Connection with Walking on a Grid



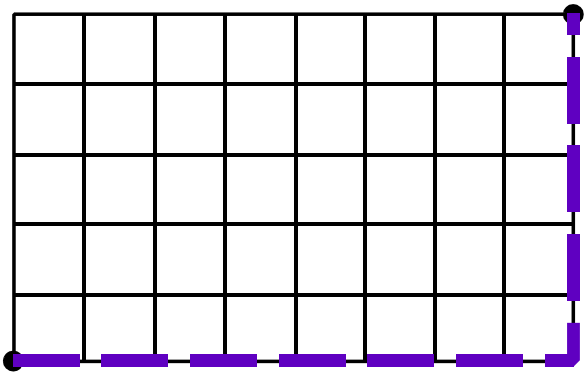
NEENENENEEEEN



EENNNEEEEEEENN



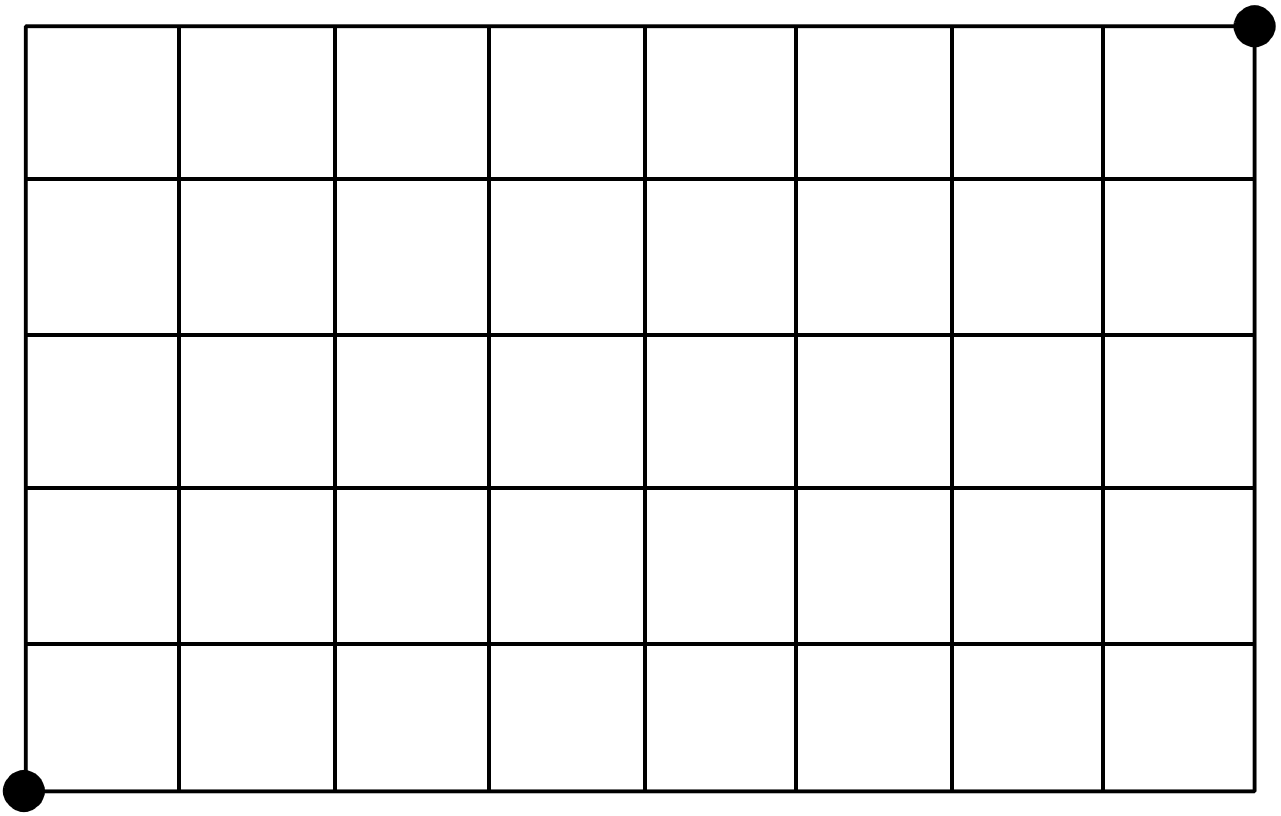
EEENENNNEENENE



EEEEEEEENNNNN

- Every way to walk takes 13 steps
- In fact, every way must have 8 steps to the east and 5 steps to the north
- Thus every way to walk corresponds to an anagram of "EEEEEEEENNNNN"

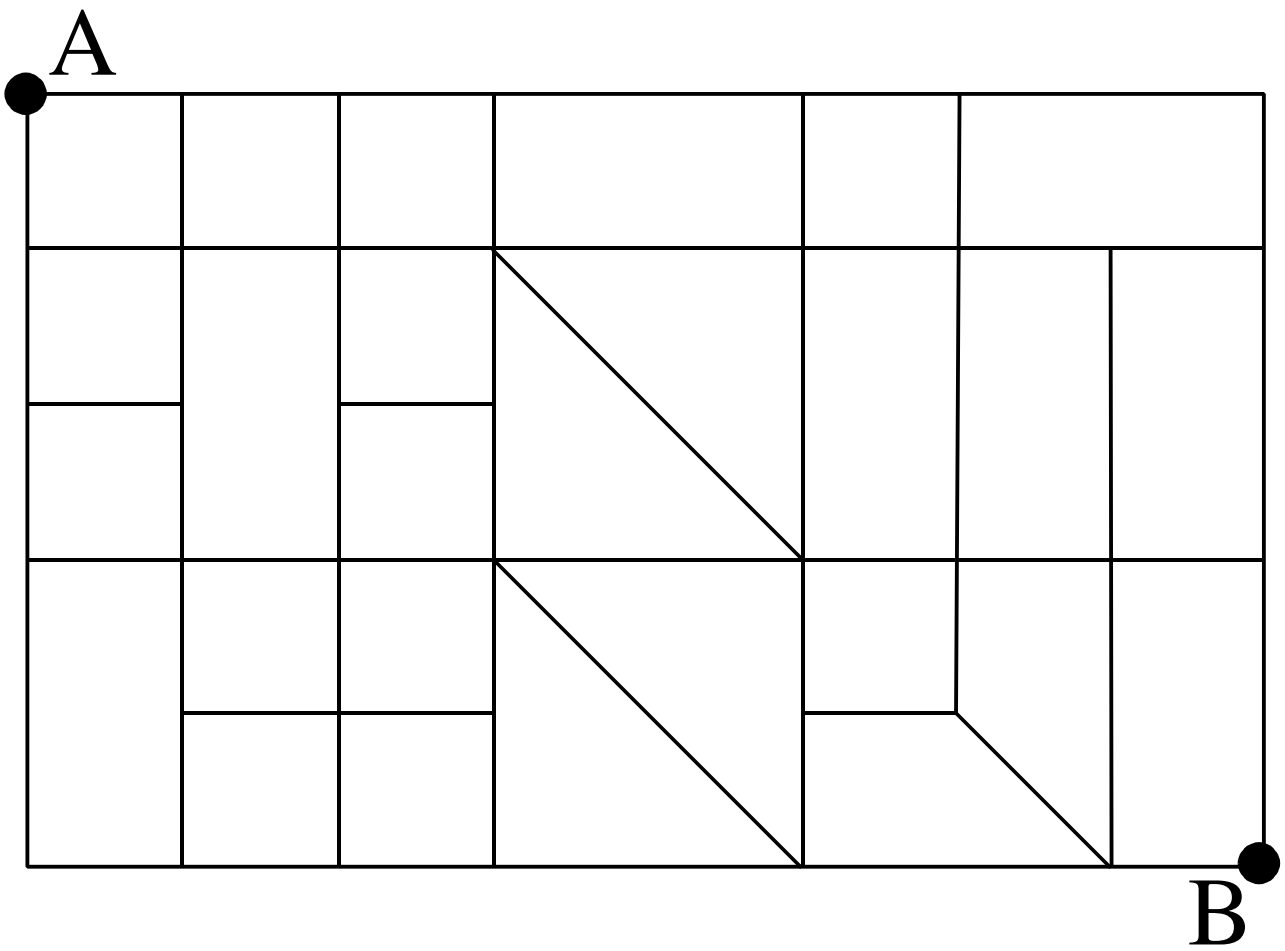
Counting Walks by Addition



The Addition Method

The addition method of counting walks may take longer, but it is much more general.

Consider the grid below. How many ways are there to walk from A to B without backtracking?



Generalized Assignment

The following "How many ways to..." questions can be thought of as *assignment* questions, in which we assign *n* distinct objects to *n* distinct positions:

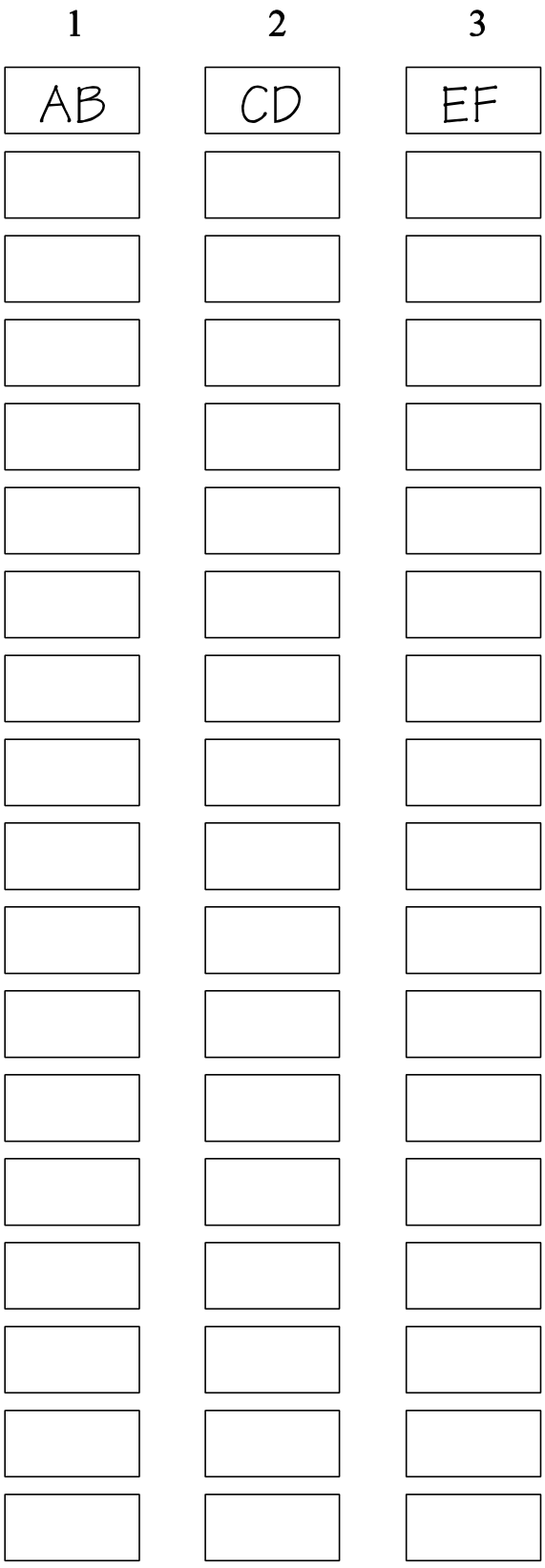
- ...assign six beds to your six children?

- ...decide which of your five employees will sweep, wash, cook, serve and bus, respectively?

- ...select the order in which you will surf, parasail, collect shells and sunbathe during your day at the beach?

How about this question:
A family has six children and three distinct bedrooms. In how many ways can the parents assign two children to each bedroom?

Assign Six Children to Three Bedrooms



Assign Three Bedrooms to Six Children

Here is one assignment of children to bedrooms:

Children	A	B	C	D	E	F
Bedrooms	1	1	2	2	3	3

Here is another:

Children	A	B	C	D	E	F
Bedrooms	1	2	3	1	2	3

And another

Children	A	B	C	D	E	F
Bedrooms	1	3	2	2	1	3

So, can you suggest the easy way?

Abstractification

Place these n Balls	Into these k Boxes	Number of Ways
distinct	distinct exactly 1 per box	$n!$
distinct	distinct n_i in the ith box	$\dfrac{n!}{n_1!\cdot\cdots\cdot n_k!}$

Practice:

There are 11 clubs which meet after school, and the club coordinator must decide which clubs should meet on which day. In how many ways can this be done if 2 clubs are to meet on each day, except for Wednesday, when 3 clubs meet.

The US Mint is releasing 5 state quarters each year for 10 years. In how many ways can this be done, assuming order of admission to the Union is not taken into account, and that the five quarters released each year are released together.

Identical Objects

I have 5 quarters. In how many ways can I place them into my two pants pockets?

Pocket 1	Pocket 2

Can a pocket be empty?

Yes: # ways =

No: # ways =

Identical Objects

What if I also use my shirt pocket, and ask that no pocket be empty?

Pocket 1	Pocket 2	Pocket 3

A Trick

P	=	pocket
❍	=	quarter

Pocket 1	Pocket 2	Pocket 3	Symbolic
2	1	2	P❍❍P❍P❍❍
0	2	3	PP❍❍P❍❍❍
5	0	0	P❍❍❍❍❍PP
2	0	3	
3	2	0	

The number of "❍"s after the *i*th "P" is the number of quarters in the *i*th pocket.

So there is a one-to-one correspondence between the ways to put quarters into pockets and anagrams of "P❍❍P❍P❍❍" *which start with "P"*, that is, with anagrams of "❍❍❍❍❍PP"

Another Trick

What if we do not allow pockets to be empty?

Then many anagrams of "P ❍❍❍❍❍P P" are not valid assignments of quarters to pockets.

The trick:

Reserve a Quarter for Each Pocket

The method:

- Take 3 quarters away
- Find anagrams, which may yield empty pockets
- Put a quarter back in each pocket

So the number of ways is equal to the number of anagrams of "❍❍PP", which is 6.

Abstractification

Place n Balls	Into these k Boxes	Number of Ways
identical	distinct boxes may be empty	$\dfrac{(n+k-1)!}{n!\cdot(k-1)!}$
identical	distinct no box empty	$\dfrac{(n-1)!}{(n-k)!\cdot(k-1)!}$

What do you think: Is it easier to remember the formula or the method?

Practice:

I passed around a stack of 25 pieces of scrap paper to my 20 students so that everyone got a piece. In how many ways could this have happened?

How many 4-digit numbers are there whose digits sum to 9?

Mixed Practice

1. How many 7-digit phone numbers are there that do not start with the digit "1"?

2. In a pretty good round of golf, Annika made 12 pars and 6 birdies. In how many ways could this have happened?

3. 30 votes were cast in an election between Xavier, Yolanda and Zoe. How many outcomes were possible?

Mixed Practice

4. How many ways are there to toss 4 dice to obtain a sum of 9, if we count 1-2-3-3 as *different* from 1-3-2-3?

5. How many ways are there to toss 4 dice to obtain a sum of 9, if we count 1-2-3-3 as *the same* as 1-3-2-3?

Deck of Cards

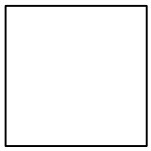
2♣	2♦	2♥	2♠
3♣	3♦	3♥	3♠
4♣	4♦	4♥	4♠
5♣	5♦	5♥	5♠
6♣	6♦	6♥	6♠
7♣	7♦	7♥	7♠
8♣	8♦	8♥	8♠
9♣	9♦	9♥	9♠
10♣	10♦	10♥	10♠
J♣	J♦	J♥	J♠
Q♣	Q♦	Q♥	Q♠
K♣	K♦	K♥	K♠
A♣	A♦	A♥	A♠

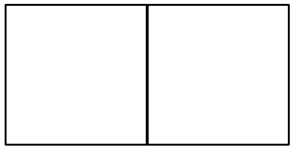
Deck of Cards

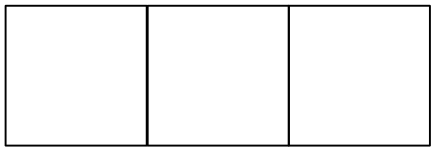
2♣	2♦	2♥	2♠
3♣	3♦	3♥	3♠
4♣	4♦	4♥	4♠
5♣	5♦	5♥	5♠
6♣	6♦	6♥	6♠
7♣	7♦	7♥	7♠
8♣	8♦	8♥	8♠
9♣	9♦	9♥	9♠
10♣	10♦	10♥	10♠
J♣	J♦	J♥	J♠
Q♣	Q♦	Q♥	Q♠
K♣	K♦	K♥	K♠
A♣	A♦	A♥	A♠

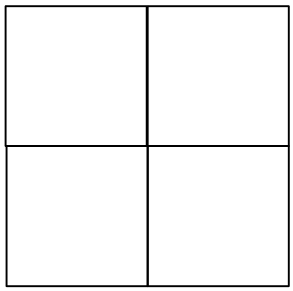
Handout #1 — Walking on Some Small Grids

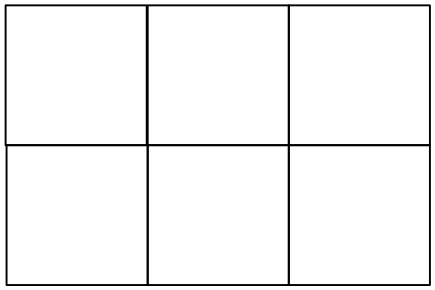
How many ways are there to walk form the top right to the bottom left corner in each of these grids?











Handout #2 — Counting Anagrams

How many anagrams are there of each of the following words?

BETTER

REPAPER

SASSY

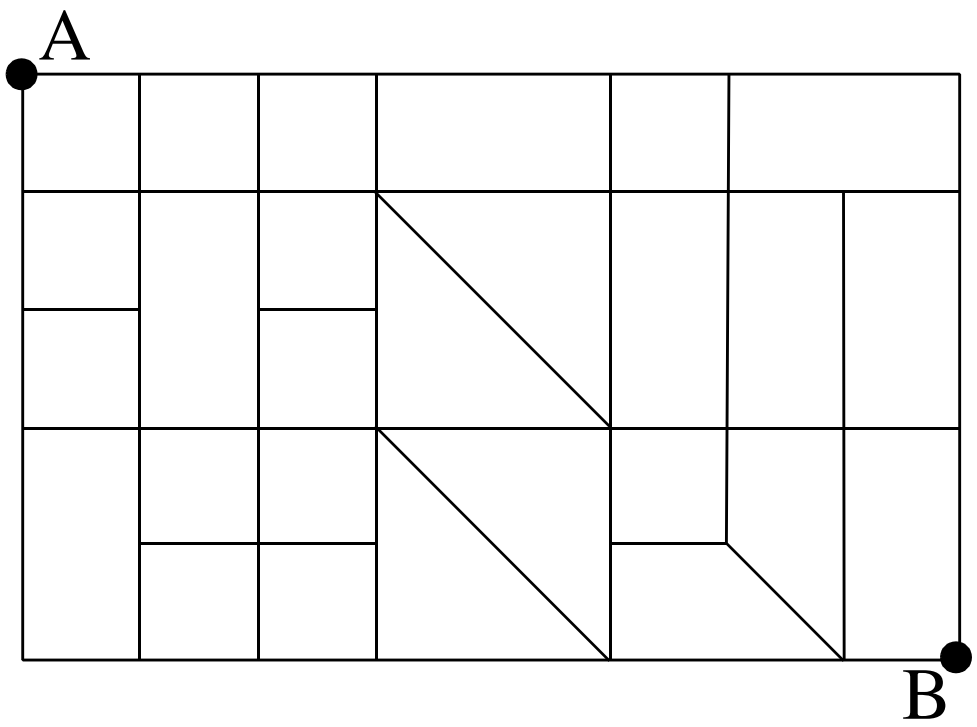
POWWOW

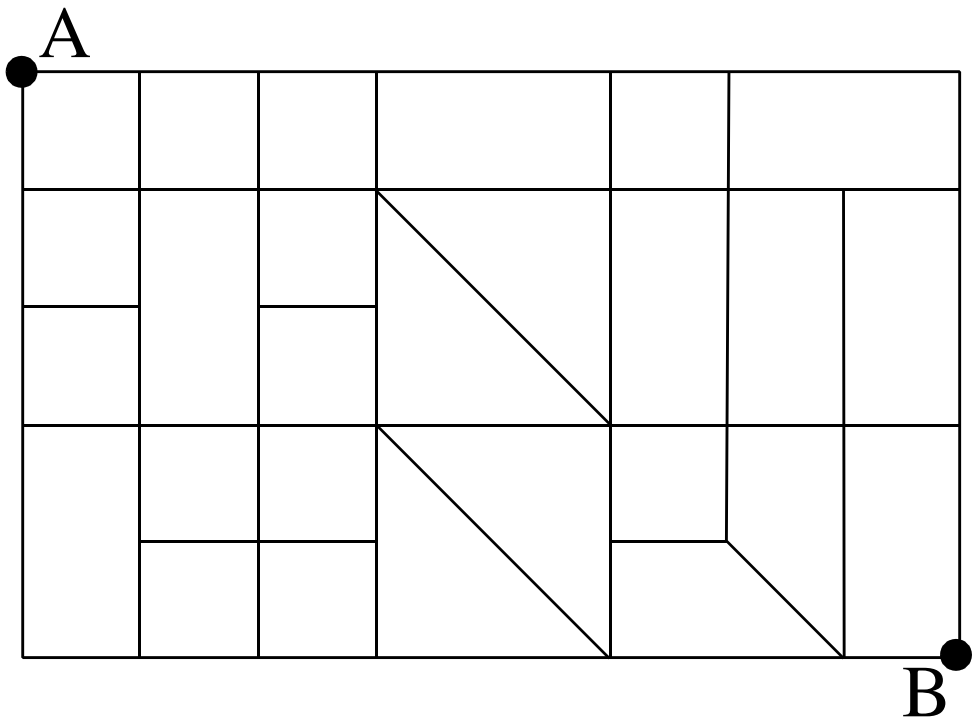
ITERATE

ANAGRAMS

Handout #3 — Walking on a Funky Grid

How many ways are there to walk from A to B in the grid below, without backtracking? (Two copies of the graph are provided.)





Handout #4 — Assigning Children to Bedrooms

How many ways are there to assign six children to three distinct bedrooms so that each bedroom gets two children? One way is shown in the first row of the chart below.

1	2	3
AB	CD	EF

How about assigning bedrooms to children:

A	B	C	D	E	F
1	1	2	2	3	3

Handout #4 — Generalized Assignment

1. There are 11 clubs which meet after school, and the club coordinator must decide which clubs should meet on which day. In how many ways can this be done if 2 clubs are to meet on each day, except for Wednesday, when 3 clubs meet.

2. The US Mint is releasing 5 state quarters each year for 10 years. In how many ways can this be done, assuming admission to the Union is not taken into account, and that the five quarters released each year are released together.

Handout #5 — Identical Balls into Distinct Boxes

1. I passed around a stack of 25 pieces of scrap paper to my 20 students so that everyone got a piece. In how many ways could this have happened?

2. How many 4-digit numbers are there whose digits sum to 9?

Handout #6 — Practice with Counting

These problems are relatively straightforward, once the structure of the problem has been determined. That tends to be the tricky part. Warning, there is one problem here we don't have a technique for.

1. How many 7-digit phone numbers are there that do not start with the digit "1"?

2. In a pretty good round of golf, Annika made 12 pars and 6 birdies. In how many ways could this have happened?

3. 30 votes were cast in an election between Xavier, Yolanda and Zoe. How many outcomes were possible?

4. How many ways are there to toss 4 dice to obtain a sum of 9, if we count 1-2-3-3 as *different* from 1-3-2-3?

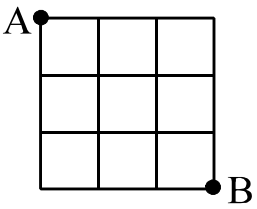
5. How many ways are there to toss 4 dice to obtain a sum of 9, if we count 1-2-3-3 as *the same* as 1-3-2-3?

Exercises — Anagrams

Warm-up Problems:

1. How many ways are there to make a 3-letter codeword from the letters of the word "AEGILOPS"? ("Aegilops" is the longest word in the English language with all letters in alphabetical order.)

2. How many ways are there to walk from A to B on the grid to the right, without backtracking?



3. How many anagrams are there of each word in this problem?

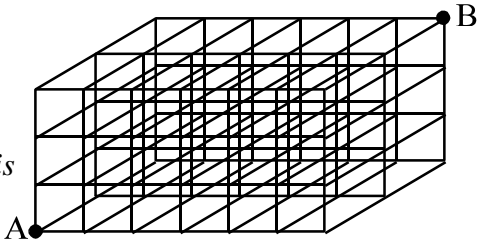
4. How many ways are there to distribute 10 identical balls into 4 distinct boxes?

5. In the grid from problem 2, there are four intersections which are exactly three steps from A. Altogether, how many ways are there to walk to *any one* of these four intersections? Can you see a way to do it that does not involve the summation of four numbers?

Presentation Problems:

6. There are 10 employees at a local store, and the manager wishes to assign 1 of them to be the cashier, 2 of them to be cleaners, 3 of them to be "stockers," and 4 of them to be salesmen. In how many ways can this be done?

7. How many ways are there to walk from A to B in the figure to the right, without backtracking? *Note: The figure represents a rectangular lattice, and there is a nice way to use anagrams for this problem.*



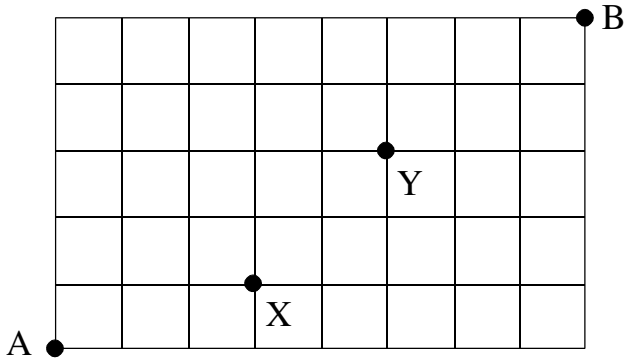
8. Four members (A, B, C and D — names not released since they are minors) of the Chewing Gum Gang are caught shoplifting. When they empty their pockets it is found that they had, between them, 20 packs of Trident and 12 packs of Dentyne. One way this could have happened is if they each had 5 packs of Trident and 3 packs of Dentyne. Another way is if A had all the Trident and B had all the Dentyne. Altogether, how many ways could the gum have been distributed?

9. Four children each grab some cookies from the cookie jar on the on their way out of the house in the morning. A total of 10 cookies were taken. In how many ways could this have happened?

10. A house has three large bedrooms and six identical beds. In how many ways can the beds be placed into the bedrooms? And how many positive integral solutions are there to $x + y + z = 30$?

11. The National Science Foundation has enough money to support fellowships for 12 graduate students, and 7 different institutions have applied for this support. The committee has now to decide how many fellowships to assign to each applying institution. In how many ways can this assignment be made so that at least one institution *does not* get a fellowship?

12. The following questions pertain to the figure to the right, where no backtracking is allowed:

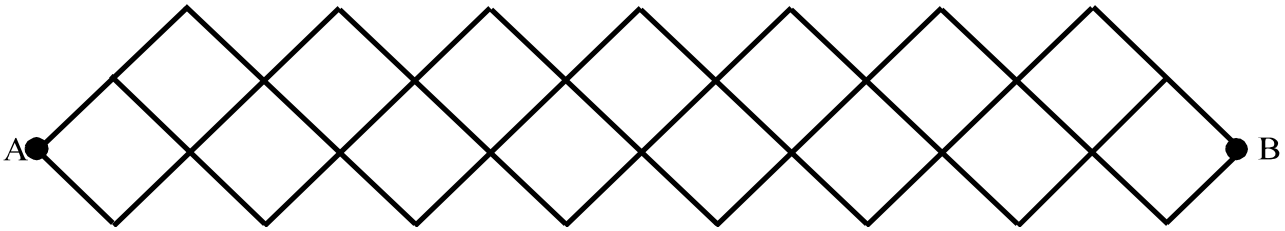


   a. How many ways are there to walk from A to B which pass through X?
   b. How many ways are there to walk from A to B which pass through both X and Y?
   c. How many ways are there to walk from A to B which *do not* pass through X?
   d. How many ways are there to walk from A to B which pass through neither X nor Y?
   e. How many ways are there to walk from A to B which pass through X but not Y?

Extension Problems:

13. A die is tossed 3 times, giving a result of 1, 2, 3, 4, 5 or 6 each time. What is the probability that each roll is higher than the previous roll? (*The answer is 5/54.*)

14. How many ways are there to walk from A to B on the figure below, without backtracking?



15. The fellow shown to the right is walking on an cliff which has 4 spaces, and he starts on the second space. He takes steps based on the flip of a coin (heads = "go right", tails = "go left") until he falls off the cliff. What is the probability that
   a. he is still safe and sound after the 4^{th} toss?
   b. he is still safe and sound after the 10^{th} toss?
   c. he falls off the cliff on exactly the 14^{th} toss?