

Discrete Mathematics --- Day 27 --- November 3, 2003

Combinatorial Identities

How many subsets does an n -element set have?

Suppose set = $\{1,2,3,4,\dots,n\}$

To build a subset, think of it as a task with n parts. For each element, decide whether it is in the subset or not.

N parts; each has 2 choices (ways to be done) total number of ways is $2*2*2*2*\dots*2 = 2^n$

Another way to answer:

Possible subset sizes?

Szes	Number of subsets of that size
0	$(nC0) 1\emptyset$
1	$(nC1) n$
2	$(nC2)$
3	$(nC3)$
.	.
.	.
.	.
n	$(nCn) 1$
Total	2^n

Thus $2^n = C(n,0) + C(n,1) + C(n,2) + \dots + C(n,n)$
 n^{th} row of Pascal's Triangle.

3rd way to answer:

What's the smallest element of each subset?

i.e. $n=4 \{1,2,3,4\}$

$\{\emptyset\}$	$\{1\}$	$\{1,2\}$	$\{1,2,3\}$	$\{1,2,3,4\}$
	$\{2\}$	$\{1,3\}$	$\{1,2,4\}$	
	$\{3\}$	$\{1,4\}$	$\{1,3,4\}$	
	$\{4\}$	$\{2,3\}$	$\{2,3,4\}$	
		$\{2,4\}$		
		$\{3,4\}$		
1	4	6	4	1

Sets with 4 as the smallest element: {4}
 Sets with 3 as the smallest element: {3} and {3,4}
 Sets with 2 as the smallest element: {2} {2,3} {2,4} and {2,3,4}
 Sets with 1 as the smallest element:
 {1} {1,2} {1,3} {1,4} {1,2,3} {1,2,4} {1,3,4} and {1,2,3,4}

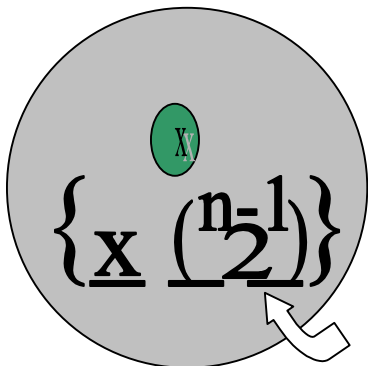
k	number of sets which have k as smallest element
1	8
2	4
3	2
4	1
total	15= 2 ⁴ -1

In General, how many subsets of {1,2,3,...,n} have x as the smallest element?
 {1,2ⁿ⁻¹}

1	2 ⁿ⁻¹	
2	2 ⁿ⁻²	{2, 2 ⁿ⁻² }
3	2 ⁿ⁻³	{2, 2 ⁿ⁻³ }
n-1	2 ⁿ⁻⁽ⁿ⁻¹⁾	
n	2 ⁿ⁻ⁿ	
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total	2 ⁿ⁻¹	

The number of ways to build a subset {1,2,...,n} that has k as the smallest element is 2^{n-k}, because we must place k in the sets and then we can add any subset of {k+1,k+2,...n(=k+n-k) } and there are 2^{n-k} such subsets, Thus: 1+2+4+8+...+2ⁿ⁻¹=2ⁿ⁻¹ -1 (“minus 1” because the empty set is missing from this list, as it has no smallest element.)

Prove that $\binom{n}{3} = \binom{n-1}{2} + \binom{n-1}{3}$



How many ways are there to select three elements from a set with n elements?

How many sets of size three don't have x?
 C(n-1,3)

How many subsets of size three do contain 'x'?
 C(n-1, 2)

Total number of subsets of size three is: $\binom{n-1}{2} + \binom{n-1}{3} = \binom{n}{3}$.